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ECS 20: Discrete Mathematics
Finals
December 16, 2014

Notes:

1) Finals are open book, open notes.
2) You have two hours, no more: It will be strictly enforced.
3) You can answer directly on these sheets (preferred), or on loose paper.
4) Please write your name at the top right of each page you turn in!
5) There are 3 parts with a total possible number of points of 110, and one extra credit problem (10 points).

Part I: proofs (4 questions; each 10 points; total 40 points)

1) Let a and b be two real numbers with $a \geq 0$ $b \geq 0$. Use a proof by contradiction to show that $\frac{a+b}{2} \geq \sqrt{ab}$.

2) Let x and y be two integers. Show that if x^2+y^2 is even, then $x+y$ is even.

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3) Let A = {1,2,3} and R = {(2,3),(2,1)} be two sets. Note that in the definition of R, order matters: for example, $(2,3) \in R$ but $(3,2) \notin R$. Prove that if *a, b*, and *c* are three elements of *A* such that $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

4) Prove or disprove that if *n* is odd, then n^2+4 is a prime number.

Part II. Proof by induction (4 questions; each 10 points; total 40 points)

1) Show by induction that $\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$ for all integer $n \geq 1$

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2) Let $\{a_n\}$ be a sequence with first terms $a_1=2$, $a_2=8$, and recursive definition:

   $a_n = 2a_{n-1} + 3a_{n-2} + 4$. Show that $a_n = 3^n - 1$ for all integer $n \geq 1$ using strong induction.

3) Let x be a positive real number (x>0). Show that $(1+x)^n > 1 + nx$ for all integer number $n > 1$.

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4) Let f_n be the n-th Fibonacci number (note: Fibonacci numbers satisfy $f_0=0, f_1=1$ and $f_n + f_{n+1} = f_{n+2}$).

Prove by induction that for all $n \geq 1$, $f_{n-1} f_{n+1} = f_n^2 + (-1)^n$.

Part III. Set theory - Functions (3 problems; each 10 points; total 30 points)

1) Let f, g, and h be three functions from R^+ to R^+ (positive real numbers). Using the definition of big-O, show that if f is $O(g(x))$ and g is $O(h(x))$, then f is $O(h(x))$.

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2) Let *A* and *B* be two sets in a universe *U*. Show that $|\overline{A} \cap \overline{B}| = |U| - |A| - |B| + |A \cap B|$ (where $|A|$ is the cardinality of *A*, i.e. number of elements in *A*).

3) Show that if n is an odd integer, then $\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2 + 3}{4}$

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Part IV. Extra credit (10 points)

Use the method of proof by **strong** induction to show that any amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps. *For example, 29 can be represented with five 5-cent stamps, and one 4-cent stamp. 17 can be represented with three 4-cent stamps, and one 5-cent stamp.*

Provide a second proof, using the method of proof by induction only (i.e. not strong induction).