Final: Solutions

ECS20 (Fall 2014)

December 16, 2014

Part I: Proofs

1) Let a and b be two real numbers with $a \geq 0$ and $b \geq 0$. Use a proof by contradiction to show that $\frac{a+b}{2} \geq \sqrt{ab}$.

   Let P be the proposition $\frac{a+b}{2} \geq \sqrt{ab}$. The concept of proof by contradiction is to assume that P is false.

   Then $\frac{a+b}{2} < \sqrt{ab}$. Raising both sides to the power 2, we get:

   $\frac{(a+b)^2}{4} < ab$

   which can be rewritten as:

   $(a+b)^2 - 4ab < 0$

   However, $(a+b)^2 - 4ab = (a-b)^2$, and this expression is positive. We have therefore reached a contradiction. The property P is therefore true. (this property in fact states that the arithmetic mean of two numbers is bigger or equal to the geometric mean of the same numbers).

2) Let x and y be two integers. Show that if $x^2 + y^2$ is even, then $x + y$ is even.

   Let p be the proposition $x^2 + y^2$ is even, and let q be the proposition $x + y$ is even. We will use an indirect proof, i.e. we will show that $\neg q \to \neg p$.

   Hypothesis: $\neg q$ is true, i.e. $x + y$ is odd. There exists an integer number k such that $x + y = 2k + 1$. Then:

   $(x+y)^2 = (2k+1)^2$

   i.e.

$$
\begin{aligned}
x^2 + y^2 + 2xy &= 4k^2 + 4k + 1 \\
x^2 + y^2 &= 4k^2 + 4k + 1 - 2xy \\
x^2 + y^2 &= 2(2k^2 + 2k - xy) + 1
\end{aligned}
$$

   Therefore $x^2 + y^2$ is odd, i.e. $\neg p$ is true.

   We can then conclude that $p \to q$ is true.

3) Let $A = \{1, 2, 3\}$ and $R = \{(2, 3), (2, 1)\}$. Prove that if a, b, and c are three elements of A such that $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

   Let P be the proposition a, b, and c are three elements of A such that $(a, b) \in R$ and $(b, c) \in R$ and let Q be the proposition $(a, c) \in R$. We want to show $P \to Q$.

   Let us study P. Since $(a, b) \in R$, $b = 3$ or $b = 1$. Since $(b, c) \in R$, $b = 2$. This two statements cannot be true at the same time: we have reached a contradiction and P is always false. If P is always false, $P \to Q$ is always true!

4) Prove or disprove that if n is odd, then $n^2 + 4$ is a prime number.

This is most likely false. We try several values of n:

- $n = 1$. Then $n^2 + 4 = 5$ that is prime.
- $n = 3$. Then $n^2 + 4 = 13$ that is prime.
- $n = 5$. Then $n^2 + 4 = 29$ that is prime.
- $n = 7$. Then $n^2 + 4 = 53$ that is prime.
- $n = 9$. Then $n^2 + 4 = 85$ that is not prime!

We have found one counter-example ($n = 9$) for which the property is not true.

Part II: Proof by induction

Exercise 1

Let $P(n)$ be the proposition:

$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$$

We want to show that $P(n)$ is true for all $n \geq 1$.

Let us define: $LHS(n) = \sum_{i=1}^{n} \frac{1}{2^i}$ and $RHS(n) = \frac{1}{2^n} - 1$.

- *Basis step*:

$$LHS(1) = \frac{1}{2} \qquad RHS(1) = 1 - \frac{1}{2} = \frac{1}{2}$$

Therefore $P(1)$ is true.

- *Induction step*: We suppose that $P(k)$ is true, with $1 \leq k$. We want to show that $P(k+1)$ is true.

$$\begin{aligned} LHS(k+1) &= \sum_{i=1}^{k+1} \frac{1}{2^i} \\ &= \sum_{i=1}^{k} \frac{1}{2^i} + \frac{1}{2^{k+1}} \\ &= LHS(k) + \frac{1}{2^{k+1}} \\ &= RHS(k) + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^{k+1}} \end{aligned}$$

and

$$RHS(k+1) = 1 - \frac{1}{2^{k+1}}$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n > 0$.

Exercise 2

Let $\{a_n\}$ be a sequence with first terms $a_1 = 2$, $a_2 = 8$, and recursive definition: $a_n = 2a_{n-1} + 3a_{n-2} + 4$.

Let $P(n)$ be the proposition:

$$a_n = 3^n - 1$$

We want to show that $P(n)$ is true for all $n \geq 1$ using a method of proof by strong induction.

Let us define: $LHS(n) = a_n$ and $RHS(n) = 3^n - 1$.

- *Basis step*:

$$LHS(1) = 2 \qquad RHS(1) = 3 - 1 = 2$$
$$LHS(2) = 8 \qquad RHS(2) = 3^2 - 1 = 9 - 1 = 8$$

  Therefore $P(1)$ and $P(2)$ are true (note that the prove that both P(1) and P(2) are true, so that we can assume $k \geq 2$ in the induction step).

- *Strong induction step*: We suppose that $P(1)$, $P(2)$, $\ldots$,$P(k)$ are true, with $2 \leq k$. We want to show that $P(k+1)$ is true.

$$\begin{aligned}
LHS(k+1) &= a_{k+1} \\
&= 2a_k + 3a_{k-1} + 4 \\
&= 2LHS(k) + 3LHS(k-1) + 4 \\
&= 2RHS(k) + 3RHS(k-1) + 4 \\
&= 2(3^k - 1) + 3(3^{k-1} - 1) + 4 \\
&= 2 \times 3^k + 3^k - 2 - 3 + 4 \\
&= 3 \times 3^k - 1 \\
&= 3^{k+1} - 1
\end{aligned}$$

  and

$$RHS(k+1) = 3^{k+1} - 1$$

  Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by strong induction allows us to conclude that $P(n)$ is true for all $n > 0$.

Exercise 3

Let x be a positive real number ($x > 0$ and let $P(n)$ be the proposition:

$$(1+x)^n > 1+nx$$

We want to show that $P(n)$ is true for all $n \geq 2$.

Let us define: $LHS(n) = (1+x)^n$ and $RHS(n) = 1+nx$.

- *Basis step*:

  $$LHS(2) = (1+x)^2 = 1+2x+x^2 \qquad RHS(2) = 1+2x$$

  Since $x > 0$, $x^2 > 0$ and $1+2x+x^2 > 1+2x$. Therefore $LHS(2) > RHS(2)$, i.e. $P(2)$ is true.

- *Induction step*: We suppose that $P(k)$ is true, with $2 \leq k$. We want to show that $P(k+1)$ is true.

  Since $P(k)$ is true, we know that:

  $$(1+x)^k > 1+kx$$

  Since x ¿ 0, 1+x ¿0. We can multiply both sides of the inequality without changing the direction:

  $$(1+x)^{k+1} > (1+x)(1+kx)$$

  We recognize $LHS(k+1)$ on the left side of this inequality:

  $$\begin{aligned} LHS(k+1) &> (1+x)(1+kx) \\ &> 1+kx+x+kx^2 \\ &> 1+(k+1)x+kx^2 \end{aligned}$$

  Since $kx^2 > 0$, $1+(k+1)x+kx^2 > 1+(k+1)x$. Therefore

  $$\begin{aligned} LHS(k+1) &> 1+(k+1)x \\ &> RHS(k+1) \end{aligned}$$

  Therefore $LHS(k+1) > RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n > 0$.

Exercise 4

Let $P(n)$ be the proposition: $f_{n-1}f_{n+1} = f_n^2 + (-1)^n$.

We define $LHS(n) = f_{n-1}f_{n+1}$ and $RHS(n) = f_n^2 + (-1)^n$. We want to show that $P(n)$ is true for all $n \geq 1$.

- *Basic step*:

$$\begin{aligned} LHS(1) &= f_0 \times f_2 = f_0 \times (f_1 + f_0) = 0 \times (1+0) = 0 \\ RHS(1) &= f_1^2 + (-1)^1 = 1 - 1 = 0 \end{aligned}$$

  Therefore $LHS(1) = RHS(1)$ and $P(1)$ is true.

- *Inductive step:* Let k be a positive integer greater or equal to 2, and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.
  Then

$$\begin{aligned} LHS(k+1) &= f_k f_{k+2} \\ &= f_k \left(f_k + f_{k+1}\right) \\ &= f_k^2 + f_k f_{k+1} \end{aligned}$$

  Using the fact that $P(k)$ is true, i.e. $f_k^2 = f_{k-1}f_{k+1} - (-1)^k$, we get:

$$\begin{aligned} LHS(k+1) &= f_{k-1}f_{k+1} - (-1)^k + f_k f_{k+1} \\ &= f_{k-1}f_{k+1} + f_k f_{k+1} - (-1)^k \\ &= \left(f_{k-1} + f_k\right) f_{k+1} + (-1)^{k+1} \\ &= f_{k+1} f_{k+1} + (-1)^{k+1} \\ &= f_{k+1}^2 + (-1)^{k+1} \end{aligned}$$

  and

$$RHS(k+1) = f_{k+1}^2 + (-1)^{k+1}$$

  Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 2$.

Part III: sets- functions

Exercise 1

Let f, g, and h be three functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. Using the definition of big-O, show that if f is $O(g)$ and g is $O(h)$ then f is $O(h)$.

By definition of big-O, f is $O(g)$ means:

$$\exists k1 > 0, \exists C1 > 0, \forall x > k1, |f(x)| \leq C1|g(x)|$$

Similiarly, g if $O(h)$ means

$$\exists k2 > 0, \exists C2 > 0, \forall x > k2, |g(x)| \leq C2|h(x)|$$

Let $k = \max(k1, k2)$. Then, for all $x > k$, we have $|f(x)| \leq C1|g(x)|$ and $|g(x)| \leq C2|h(x)|$, therefore $|f(x) \leq C1C2|h(x)|$.

Let $C = C1C2$. We have found that:

$$\exists k > 0, \exists C > 0, \forall x > k, |f(x)| \leq C|h(x)|$$

Therefore f if $O(h)$.

Exercise 2

Let A, B, and C be three sets in a universe U. Show that $|\overline{A}\bigcap\overline{B}| = |U| - |A| - |B| + |A\bigcap B|$.

We note first that according to deMorgan's law,

$$\overline{A}\bigcap\overline{B} = \overline{AUB}$$

Based on the complement's law,

$$|\overline{A}\bigcap\overline{B}| = |\overline{AUB}| = |U| - |A\bigcup B|$$

Finally, based on the inclusion-exclusion principle,

$$|A\bigcup B| = |A| + |B| - |A\bigcap B|$$

Replacing in the equation above, we get:

$$|\overline{A}\bigcap\overline{B}| = |U| - |A| - |B| + |A\bigcap B|$$

i.e. the property is true.

Exercise 3

Show that if n is an odd integer, $\lceil \frac{n^2}{4} \rceil = \frac{n^2+3}{4}$

We use a direct proof. Let n be an odd integer and let us define $LHS(n) = \lceil \frac{n^2}{4} \rceil$ and $RHS(n) = \frac{n^2+3}{4}$.

Let n be an odd integer There exists $k \in \mathbb{Z}$ such that $n = 2k+1$. Then $n^2 = 4k^2 + 4k + 1$. Therefore:

$$\begin{aligned} LHS(n) &= \left\lceil \frac{4k^2+4k+1}{4} \right\rceil \\ &= \left\lceil k^2 + k + \frac{1}{4} \right\rceil \\ &= k^2 + k + \left\lceil \frac{1}{4} \right\rceil \\ &= k^2 + k + 1 \end{aligned}$$

and

$$\begin{aligned} RHS(n) &= \frac{4k^2+4k+1+3}{4} \\ &= \frac{4k^2+4k+4}{4} \\ &= k^2 + k + 1 \end{aligned}$$

Therefore $LHS(n) = RHS(n)$. The property is true.

Extra credit

Use the method of proof by strong induction to show that any amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Let $P(n)$ be the property: the amount of postage of n cents can be formed using just 4-cent and 5-cent stamps. We want the show that $P(n)$ is true, for all $n \geq 12$.

Let us first analyze what this property means. We can rewrite it as: "There exists two non-negative integers m and p such that $n = 4m + 5p$. We prove the property first using strong induction.

- *Basis step*: We want to show that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are true.

  Note that $12 = 4 \times 3 + 5 \times 0$. We found a pair of non negative integers $(m, p) = (3, 0)$ such that $12 = 4m + 5p$. P(12) is therefore true. Note that $13 = 4 \times 2 + 5 \times 1$. We found a pair of non negative integers $(m, p) = (2, 1)$ such that $13 = 4m + 5p$. P(13) is therefore true. Note that $14 = 4 \times 1 + 5 \times 2$. We found a pair of non negative integers $(m, p) = (1, 2)$ such that $14 = 4m + 5p$. P(14) is therefore true. Note that $15 = 4 \times 0 + 5 \times 3$. We found a pair of non negative integers $(m, p) = (0, 3)$ such that $15 = 4m + 5p$. P(14) is therefore true.

- *Strong induction step*: We suppose that $P(12), P(13), \ldots,$ and $P(k)$ are true, for $k \geq 15$, and we want to show that $P(k+1)$ is true.

  Since P is true for all values up to k, it is true in particular for $k - 3$ (we are allowed to use $k - 3$ as $k \geq 15$. Therefore, there exists two non negative integers (m, p) such that

  $$k - 3 = 4m + 5p$$

  Adding 4 to this equation, we get:

  $$k + 1 = 4(m + 1) + 5p$$

  We found a pair of non negative integers $(m', p') = (m + 1, p)$ such that $k + 1 = 4m' + 5p'$. P(k+1) is therefore true.

The principle of proof by strong induction allows us to conclude that $P(n)$ is true for all $n \geq 12$.

Let us repeat the proof, but this time we only use induction.

- *Basis step*: It remains the same. We want to show that $P(12)$ is true.

  Note that $12 = 4 \times 3 + 5 \times 0$. We found a pair of non negative integers $(m, p) = (3, 0)$ such that $12 = 4m + 5p$. P(12) is therefore true.

- *induction step*: This time, we only suppose that $P(k)$ is true, for $k \geq 12$, and we want to show that $P(k+1)$ is true.

  Since $P(k)$ is true, there exists two non negative integers (m, p) such that

  $$k = 4m + 5p$$

  Adding 1 to this equation, we get:

  $$k + 1 = 4m + 5p + 1$$

We notice that 1 can be written as 5 - 4. In which case:

$$\begin{aligned} k+1 &= 4m+5p+5-4 \\ &= 4(m-1)+5(p+1) \end{aligned}$$

$m-1$ may not be non-negative however, based on the value of m. We therefore distinguish two cases:

- $m \neq 0$ In this case, $m-1$ is non negative. We found a pair of non negative integers $(m', p') = (m-1, p+1)$ such that $k+1 = 4m' + 5p'$. P(k+1) is therefore true.
- $m = 0$ In this case, $m-1$ is negative. Let us go back to

$$\begin{aligned} k+1 &= 4m+5p+1 \\ &= 5p+1 \end{aligned}$$

Since $m = 0$. We note first that $p \geq 3$ as $k \geq 12$. We notice then that $1 = 16 - 15$. In this case:

$$\begin{aligned} k+1 &= 5p+16-15 \\ &= 4 \times 4 + 5(p-3) \end{aligned}$$

with 4 and $p-3$ being non negative. We found a pair of non negative integers $(m', p') = (4, p-3)$ such that $k+1 = 4m' + 5p'$. P(k+1) is therefore true.

In both cases, $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 12$.