

Final: Solutions

ECS20 (Fall 2016)

December 9, 2016

1 Part I: logic (2 questions, each 10 points; total 20 points)

- 1) For each proposition on the left, indicate if it is a tautology or not:

Table 1: Propositional logic

Proposition	Tautology (Yes/ No)
if $1+3=5$ then $3=6$	Yes! This is $p \rightarrow q$ where p is false: therefore $p \rightarrow q$ is always true
$(p \wedge \neg p) \rightarrow q$	Yes: since $p \wedge \neg p$ is always false, this is of the form $A \rightarrow q$ where A is false: therefore $A \rightarrow q$ is always true
$(p \vee \neg p) \rightarrow (q \wedge \neg q)$	No: notice that $p \vee \neg p$ is always true and $q \wedge \neg q$ is always false, then this is of the form $T \rightarrow F$ which is always false
if $2+2=4$ then $3=4$	No! This is $p \rightarrow q$ where p is true and q is false, therefore $p \rightarrow q$ is false
$(p \wedge \neg q) \vee (\neg p \vee q)$	Yes! If $A = p \wedge \neg q$ then this is of the form $A \vee \neg A$ which is always true.

- 2) Lady and Tiger

Let us play a logical game. You find yourself in front of two rooms whose doors are closed. If a lady is in Room I, then the sign on the door is true, but if a tiger is in it, the sign is false. In Room II, the situation is the opposite: a lady in the room means the sign on the door is false, and a tiger in the room means the sign is true. It is possible that both rooms contain ladies or both rooms contain tigers, or that one room contains a lady and the other a tiger. Both signs say "both rooms contain ladies. Can you say what each room contains? Explain your answer.

Let us build the table for the possible options for room I and room II. We then check the validity of the two signss, and finally check the consistency of the truth values for those statements with what is told to us about the rooms.

Therefore room I contains a Tiger, and room II contains a Lady.

room I	room II	Sign on room I	sign on room II	Compatibility
Lady	Lady	True	True	False for room II: contains a Lady, but sign is true
Lady	Tiger	False	False	False for room I: contains a Lady, but sign is false
Tiger	Lady	False	False	Consistent
Tiger	Tiger	False	False	False for room II: contains a tiger, but sign is false

2 Part II: Proofs (4 questions, each 10 points; total 40 points)

- 1) Let a and b be two strictly positive real numbers. Use a proof by contradiction to show that if $\frac{a}{b+1} = \frac{b}{a+1}$ then $a = b$.

This is a problem of showing that a conditional $p \rightarrow q$ is true, where

$$p : \frac{a}{b+1} = \frac{b}{a+1}$$

$$q : a = b$$

We will use a proof by contradiction

Assumption: we suppose $p \rightarrow q$ is false i.e. that p is true AND $\neg q$ is true, namely $\frac{a}{b+1} = \frac{b}{a+1}$ AND $a \neq b$.

$\frac{a}{b+1} = \frac{b}{a+1}$ can be rewritten $a(a+1) = b(b+1)$, i.e. $a^2 - b^2 = b^2 - a^2$, i.e. $a - b = (b - a)(b + a)$. Since we suppose that $a \neq b$, we can divide by $(a - b)$ and we get that $b + a = -1$. However, a and b are strictly positive real numbers; we have reached a contradiction.

Therefore $p \rightarrow q$ is true, i.e. if $\frac{a}{b+1} = \frac{b}{a+1}$ then $a = b$.

- 2) Let n be a natural number. Show that $n(n+1)$ is divisible by 2.

This is a very simple problem that is solved using a proof by case. Let n be a natural number. There are two cases to consider:

- a) n is even. There exists an integer k such that $n = 2k$. Then $n(n+1) = 2k(2k+1)$ is a multiple of 2, therefore divisible by 2.
- b) n is odd. There exists an integer k such that $n = 2k+1$. Then $n(n+1) = (2k+1)(2k+2) = 2(k+1)(2k+1)$ is a multiple of 2, therefore divisible by 2.

In all cases, $n(n+1)$ is divisible by 2.

- 3) Let A , B , and C be three sets in a domain D . Show that if $A \in B$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$

We will use a proof by contradiction.

Hypothesis: $A \in B$ and $B \cap C = \emptyset$ AND $A \cap C \neq \emptyset$.

Since $A \cap C \neq \emptyset$, there exists an x in D that belongs to $A \cap C$. x belongs to A and x belongs to C . Since x belongs to A , and $A \in B$, x belongs to B . Therefore x belongs to both B and C ; but since $B \cap C = \emptyset$, this element x does not exist. We have reached a contradiction.

Therefore the initial proposition is true.

- 4) Evaluate the remainder of the division of 14^{3141} by 17.

Let $A = 14^{3139}$. Note first that $3139 = 17 \times 184 + 11$, therefore $A = (14^{184})^{17} \times 14^{11}$. Since 17 is prime, we can use Fermat's little theorem, i.e. for all natural number a ,

$$a^{17} \equiv a[17]$$

Therefore,

$$\begin{aligned} A &\equiv (14^{184})^{17} \times 14^{11}[17] \\ &\equiv 14^{184} \times 14^{11}[17] \\ &\equiv 14^{195}[17] \end{aligned}$$

Now we note that $195 = 11 \times 17 + 8$, therefore $14^{195} = (14^{11})^{17} \times 14^8$. Therefore:

$$\begin{aligned} A &\equiv 14^{195}[17] \\ &\equiv (14^{11})^{17} \times 14^8[17] \\ &\equiv 14^{11} \times 14^8[17] \\ &\equiv 14^{19}[17] \end{aligned}$$

Now we note that $19 = 1 \times 17 + 2$, therefore $14^{19} = (14)^{17} \times 14^2$. Therefore:

$$\begin{aligned} A &\equiv 14^{19}[17] \\ &\equiv 14^{17} \times 14^2[17] \\ &\equiv 14^3[17] \end{aligned}$$

Note that $14^2 \equiv 9[17]$, therefore $14^3 \equiv 7[17]$.

Therefore

$$A \equiv 7[17]$$

and the remainder of the division of 14^{3139} by 17 is 7.

3 Part III: Proof by induction (3 questions; each 10 points; total 30 points)

- 1) Prove by induction that $3^{2n+1} + 2^{n+2}$ is divisible by 7, $\forall n \geq 0$.

Let us define $LHS(n) = 3^{2n+1} + 2^{n+2}$

Let $p(n) : 7/LHS(n)$

We want to show $p(n)$ is true for all $n \geq 0$

a) Base Case $n=0$

$$LHS(0) = 3 + 4 = 7$$

Since $7/LHS(0)$, $p(0)$ is true

b) Inductive Step

I want to show $p(k) \rightarrow p(k+1)$ whenever $k \geq 1$

$p(k)$ is true means there exists an integer m such that $LHS(k) = 3^{2k+1} + 2^{k+2} = 7m$, which we rewrite as $2^{k+2} = -3^{2k+1} + 7m$.

Note that:

$$\begin{aligned} LHS(k+1) &= 3^{2k+3} + 2^{k+3} \\ &= 9 \times 3^{2k+1} + 2 \times 2^{k+2} \\ &= 9 \times 3^{2k+1} + 2 \times (-3^{2k+1} + 7m) \\ &= 7(2m) + 7 \times 3^{2k+1} \\ &= 7(2m + 3^{2k+1}) \end{aligned}$$

Therefore $7/LHS(k+1)$ which validates that $p(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $p(n)$ is true for all $n \geq 1$.

2) **Prove by induction that $4^n - 1$ is divisible by 5 whenever n is a strictly positive even integer**

Let $p(n)$ be the proposition that $4^n - 1$ is divisible by 5.

This is not always true: it is not true for example when $n = 1$, or when $n = 3$. It is true however when n is even. But we need to modify this proposition to make it amenable to a proof by induction.

Notice that n is even and $n > 0$ means that there exists $k \in \mathbb{N}$ such that $n = 2k$. We rewrite $p(n)$ as a new proposition Q that depends on k :

$Q(k) : 4^{2k} - 1$ is divisible by 5. We need to show $Q(k)$ for all integer values $k > 0$! We are back to a traditional proof by induction.

a) Basis step $k=1$

Note that $4^2 - 1 = 16 - 1 = 15$ is divisible by 5. Therefore $Q(1)$ is true.

b) Inductive Step

We want to show $Q(k) \rightarrow Q(k+1)$ whenever $k \geq 1$

Hypothesis: $Q(k)$ is true: $4^{2k} - 1$ is divisible by 5; there exists $m \in \mathbb{Z}$ such that $4^{2k} - 1 = 5m$.

Let us compute $A = 4^{2(k+1)} - 1$:

$$\begin{aligned} A &= 4^{2k+2} - 1 \\ &= 16 \times 4^{2k} - 1 \end{aligned}$$

Replacing 4^{2k} by $5m + 1$ (based on $Q(k)$ being true), we get:

$$\begin{aligned} A &= 16(5m + 1) - 1 \\ &= 5(16m) + 15 \\ &= 5(16m + 3) \end{aligned}$$

Therefore $Q(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $Q(k)$ is true for all $k \geq 1$.

3) Let x and y be two strictly positive integers. We consider the equation:

$$(x^2 + xy - y^2)^2 = 1 \quad (1)$$

If (x, y) satisfies this equation, it is called a solution. Let now a_n be the sequence defined by $a_0 = a_1 = 1$ and $a_{n+2} = a_{n+1} + a_n$. Show by induction that (a_n, a_{n+1}) is a solution for all $n \geq 0$.

Let $P(n)$ be the proposition: (a_n, a_{n+1}) . We want to show that $P(n)$ is true for all $n \geq 0$. Let us define $L(n) = (a_n^2 + a_n a_{n+1} - a_{n+1}^2)^2$. $P(n)$ means $L(n) = 1$.

a) Basis step $n=0$

Since $a_0 = a_1 = 1$, we check if the pair $(1, 1)$ is a solution to the equation: Since $L(1) = (1 + 1 - 1)^2 = 1$, $(1, 1)$ is indeed a solution.

Therefore $P(0)$ is true.

b) Inductive Step

We want to show $P(k) \rightarrow P(k+1)$ whenever $k \geq 0$

Hypothesis: $(P(k))$ is true, i.e. a_k, a_{k+1} is a solution, i.e. $L(k) = (a_k^2 + a_k a_{k+1} - a_{k+1}^2)^2 = 1$:

Let us compute $L(k+1)$:

$$L(k+1) = (a_{k+1}^2 + a_{k+1} a_{k+2} - a_{k+2}^2)^2$$

We replace all a_{k+2} with $a_{k+1} + a_k$:

$$\begin{aligned} L(k+1) &= (a_{k+1}^2 + a_{k+1}(a_{k+1} + a_k) - (a_{k+1} + a_k)^2)^2 \\ &= (a_{k+1}^2 + a_{k+1}^2 + a_{k+1}a_k - a_{k+1}^2 - 2a_{k+1}a_k - a_k^2)^2 \\ &= (a_{k+1}^2 - a_{k+1}a_k - a_k^2)^2 \\ &= (-L(k))^2 \\ &= 1 \end{aligned}$$

Therefore $L(k+1) = 1$ which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 0$.

4 Part IV: Counting. (2 problems; each 10 points; total 20 points)

- 1) Let $A = \{a, b, c\}$ be a set with three elements. We call a , b , and c “letters”. How many words of length n can we form with only letters from A that contain at least one of each letter from A ?

Let W be the set of words of length n formed with only letters from A . There are 3^n such words. Let B be the subset of those words that contain at least one of each letter. To find the cardinality of B , we consider instead \overline{B} , i.e. the subset of words that either do not contain a , or do not contain b , or do not contain c : $\overline{B} = \overline{B}_a \cup \overline{B}_b \cup \overline{B}_c$.

- $|\overline{B}_a| = 2^n$
- $|\overline{B}_b| = 2^n$
- $|\overline{B}_c| = 2^n$
- $|\overline{B}_a \cap \overline{B}_b| = 1$
- $|\overline{B}_a \cap \overline{B}_c| = 1$
- $|\overline{B}_b \cap \overline{B}_c| = 1$
- $|\overline{B}_a \cap \overline{B}_b \cap \overline{B}_c| = 0$

Therefore $|\overline{B}| = 3 \times 2^n - 3$, and $|B| = 3^n - 3 \times 2^n + 3$.

- 2) Show that if eleven distinct numbers are selected in the set $S = \{1, 2, ?, 20\}$, there are (at least) two of them whose difference is equal to 5.

This is a Pigeonhole problem. We need to define the pigeons and the boxes:

Pigeons: the 11 numbers

Boxes: we group the number into 10 boxes:

$$\begin{array}{llllll} B1 = \{1, 6\} & B2 = \{2, 7\} & B3 = \{3, 8\} & B4 = \{4, 9\} & B5 = \{5, 10\} & B6 = \\ \{11, 16\} & B7 = \{12, 17\} & B8 = \{13, 18\} & B9 = \{14, 19\} & B10 = \{15, 20\} & \end{array}$$

According to the PHP, since we have 11 pigeons and 10 boxes, one of these boxes will contain two elements; the difference between those two elements is 5, by construction of the box!

5 Part V: Extra credit

Let n be a strictly positive integer. Use strong induction to show that

$$\sum_{i=1}^n (-1)^i i^2 = (-1)^n \sum_{i=1}^n i$$

for all $n \geq 1$.

We define $LHS(n) = \sum_{i=1}^n (-1)^i i^2$ and $RHS(n) = (-1)^n \sum_{i=1}^n i$. Let $P(n)$ be the proposition: $LHS(n) = RHS(n)$.

We want to show that $P(n)$ is true for all $n \geq 1$.

- a) Basis cases: $n=1$ and $n=2$
 $LHS(1) = (-1) \times 1 = -1$

$$RHS(1) = (-1) \times 1 = -1$$

Therefore $LHS(1) = RHS(1)$ and $P(1)$ is true.

$$LHS(2) = (-1) \times 1 + (-1)^2 2^2 = -1 + 4 = 3$$

$$RHS(2) = (-1)^2 \times (1 + 2) = 3$$

Therefore $LHS(2) = RHS(2)$ and $P(2)$ is true.

b) Inductive Step

I want to show $[P(1) \wedge \dots P(k)] \rightarrow P(k+1)$ whenever $k \geq 2$

Hypothesis: We will only need to use $P(k-1)$ is true, i.e. $LHS(k-1) = RHS(k-1)$.

Note that:

$$\begin{aligned}
LHS(k+1) &= \sum_{i=1}^{k+1} (-1)^i i^2 \\
&= \sum_{i=1}^{k-1} (-1)^i i^2 + (-1)^k k^2 + (-1)^{k+1} (k+1)^2 \\
&= LHS(k-1) + (-1)^k k^2 + (-1)^{k+1} (k+1)^2 \\
&= RHS(k-1) + (-1)^k k^2 + (-1)^{k+1} (k+1)^2 \\
&= (-1)^{k-1} \sum_{i=1}^{k-1} i + (-1)^k k^2 + (-1)^{k+1} (k+1)^2 \\
&= (-1)^{k+1} \left[\sum_{i=1}^{k-1} i - k^2 + (k+1)^2 \right] \\
&= (-1)^{k+1} \left[\sum_{i=1}^{k-1} i - k^2 + k^2 + 2k + 1 \right] \\
&= (-1)^{k+1} \left[\sum_{i=1}^{k-1} i + k + k + 1 \right] \\
&= (-1)^{k+1} \sum_{i=1}^{k+1} i \\
&= RHS(k+1)
\end{aligned}$$

Therefore $LHS(k+1) = RHS(k+1)$ which validates that $P(k+1)$ is true.

The principle of proof by strong mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 1$.