Midterm1 (sample1): Solutions

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Patrice Koehl
koehl@cs.ucdavis.edu

Part I

Exercise 1

Let us build the Truth table for $A = (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$p \wedge \neg q$	$\neg p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (p \wedge \neg q)$	$(\neg p \wedge q) \vee (\neg p \wedge \neg q)$	A
T	T	F	F	T	F	F	F	T	F	T
T	F	F	T	F	T	F	F	T	F	T
F	T	T	F	F	F	T	F	F	T	T
F	F	T	T	F	F	F	T	F	T	T

From Column 11, we can conclude that A is a tautology.

Another approach is to use logical equivalences:

$$\begin{array}{rcll} (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) & \Longleftrightarrow & (p \wedge (q \vee \neg q) \vee (\neg p \wedge (q \vee \neg q) & \textit{distributivity} \\ & \Longleftrightarrow & (p \wedge T) \vee (\neg p \wedge T) & \textit{complement law} \\ & \Longleftrightarrow & p \vee \neg p & \textit{absorption law} \\ & \Longleftrightarrow & T & \textit{complement law} \end{array}$$

Exercise 2

We can use a truth table, but it is much easier to use logical equivalences:
Note that : $(\neg p) \vee (\neg q) \vee (\neg r) \Longleftrightarrow \neg(p \wedge q \wedge r)$.
Therefore:
$(p \wedge q \wedge r) \vee (\neg p) \vee (\neg q) \vee (\neg r) \Longleftrightarrow (p \wedge q \wedge r) \vee \neg(p \wedge q \wedge r) \Longleftrightarrow T$
hence the proposition is a tautology.

Exercise 3

Let us build the truth table for $B = \neg(p \rightarrow \neg q) \rightarrow \neg(p \leftrightarrow \neg q)$:

p	q	$\not{q}$	$(p \rightarrow \neg q)$	$\neg(p \rightarrow \neg q)$	$p \leftrightarrow \neg q$	$\neg(p \leftrightarrow \neg q)$	B
T	T	F	F	T	F	T	T
T	F	T	T	F	T	F	T
F	T	F	T	F	T	F	T
F	F	T	T	F	F	T	T

From Column 8, we can conclude that B is a tautology.

Part II

Exercise 1

Prove or disprove that if n is an odd integer, the $n^2 + 4$ is a prime number.

This proposition is probably not true, in which case we only need one counter-example to invalidate it:
Note that if $n = 11$, $n^2 + 4 = 125 = 5 * 5 * 5$, which is not prime. The proposition is false.

Exercise 2

Show that if n is an integer such that $n^2 + 4n + 3$ is odd, then n is even.
Let p be the proposition: "$n^2 + 4n + 3$ is odd", and let q be the proposition "n is even".
We want to show that $p \to q$ is true. It is however very difficult to start from p, therefore we will use an indirect proof, i.e. we prove that the contrapositive $\neg q \to \neg p$ is true.
Therefore, we want to prove: if n is odd, then $n^2 + 4n + 3$ is even.
Let n be an odd integer. There exists an integer k such that $n = 2k + 1$. Then:

$$\begin{aligned} n^2 + 4n + 3 &= (2k+1)^2 + 4 * (2k+1) + 3 \\ &= 4k^2 + 4k + 1 + 8k + 4 + 3 \\ &= 4k^2 + 12k + 8 \\ &= 2 * (2k^2 + 6k + 4) \end{aligned}$$

Therefore $n^2 + 4n + 3$ is a multiple of 2, i.e. it is even.

Exercise 3

Prove or disprove that $\forall n > 1$, there are no 3 integers x, y and z such that $x^n + y^n = z^n$.

The proposition is probably false, in which case we only need to find one counterexample.
Note that for $n = 2$, $3^2 + 4^2 = 5^2$, i.e. for $n = 2$, we found 3 integers x, y and z such that $x^2 + y^2 = z^2$.
The proposition is false.
(we could also have chosen $x = y = z = 0$).