

# Midterm1 practice 2: Solutions

ECS20 (Spring 2016)

## Part I: logic

Using truth tables, establish for each of the two propositions below if it is a tautology, a contradiction or neither.

1)  $(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q)$

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$p$	$q$	$p \leftrightarrow q$	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$	$(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T

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The proposition is a tautology.

2)  $(p \rightarrow (q \wedge r)) \vee ((p \wedge q) \rightarrow r)$

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$p$	$q$	$r$	$q \wedge r$	$p \rightarrow (q \wedge r)$	$p \wedge q$	$(p \wedge q) \rightarrow r$	$(p \rightarrow (q \wedge r)) \vee ((p \wedge q) \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	T
T	F	F	F	F	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	F	T	T
F	F	F	F	T	F	T	T

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The proposition is not a tautology.

## Part II: logic- satisfiability

### Exercise 1

A set of compound propositions is said to be satisfiable if we can assign truth-values to its variables that make each proposition true. For example, the set  $\{p \wedge q, p \vee q\}$  is satisfiable as when we assign  $p$  and  $q$  to be True, both  $p \wedge q$  and  $p \vee q$  are true.

Prove, or disprove that the set  $\{p \vee q, p \vee \neg q, \neg p \vee q, \neg p \vee \neg q \vee \neg r\}$  is satisfiable. If it is, give the possible truth-values for  $p, q$ , and  $r$ .

Let us define  $A = p \vee q$ ,  $B = p \vee \neg q$ ,  $C = \neg p \vee q$ , and  $D = \neg p \vee \neg q \vee \neg r$ . We build the truth table for these four propositions;

$p$	$q$	$r$	$\neg p$	$\neg q$	$\neg r$	$A$	$B$	$C$	$D$
T	T	T	F	F	F	T	T	T	F
T	T	F	F	F	T	T	T	T	T
T	F	T	F	T	F	T	T	F	T
T	F	F	F	T	T	T	T	F	T
F	T	T	T	F	F	T	F	T	T
F	T	F	T	F	T	T	F	T	T
F	F	T	T	T	F	F	T	T	T
F	F	F	T	T	T	F	T	T	T

Based on this truth table, if we set  $p$  to True,  $q$  to True, and  $r$  to False, all four propositions  $A, B, C$ , and  $D$  are true: the set is therefore satisfiable.

## Exercise 2

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet three inhabitants: Alex, John and Sally. Alex says, ‘At least one of the following is true: that Sally is a knave or that I am a knight.’ John says, ‘Alex could claim that I am a knave.’ Sally claims, ‘Neither Alex nor John are knights.’ Can you find who is a knight and who is a knave?

The easiest here is to use a table: K means knight, N means knave, T means that the person said the truth, and F that the person said a lie.

Line	Alex	John	Sally	Alex says	John says	Sally says
1	Knight	Knight	Knight	T	F	F
2	Knight	Knight	Knave	T	F	F
3	Knight	Knave	Knight	T	T	F
4	Knight	Knave	Knave	T	T	F
5	Knave	Knight	Knight	F	T	F
6	Knave	Knight	Knave	T	T	F
7	Knave	Knave	Knight	F	F	T
8	Knave	Knave	Knave	T	F	T

Now we note that:

- line 1 cannot be correct: John and Sally would be knights that lie.
- line 2 cannot be correct: John would be a knight that lies.

- line 3 cannot be correct: John would be a knave that tells the truth (and Sally a knight that lies).
- line 4 cannot be correct: John would be a knave that tells the truth.
- line 5 cannot be correct: Sally would be a knight that lies.
- line 6 cannot be correct: Alex would be a knave that tells the truth.
- line 8 cannot be correct: Alex would be a knave that tells the truth.

Line 7 is the only possibility, i.e. Alex and John are knaves and Sally is a knight.

## Part III: proofs

### Exercise 1

Let  $P$  be the proposition: If  $5n^2 + 2$  is even then  $n$  is even.

This is an implication of the form  $p \rightarrow q$ , with:

$p$ :  $5n^2 + 2$  is even

$q$ :  $n$  is even

where  $n$  is an integer.

We use an indirect proof, i.e. we show that  $\neg q \rightarrow \neg p$ .

*Hypothesis*:  $\neg q$  is true:  $n$  is odd. By definition, there exists  $k \in \mathbb{Z}$  such that  $n = 2k + 1$ .

Therefore:  $5n^2 + 2 = 5(2k + 1)^2 + 2 = 20k^2 + 20k + 5 + 2 = 2(10k^2 + 10k + 3) + 1$ .  $5n^2 + 2$  is of the form  $2l + 1$ ; it is therefore odd; this concludes the proof.

### Exercise 2

Let  $a, b$ , and  $c$  be three consecutive integers with  $a < b < c$ . Show that if  $a \neq -1$  and  $a \neq 3$ , then  $a^2 + b^2 \neq c^2$ .

This is an implication of the form  $p \rightarrow q$ , with:

$p$ :  $a \neq -1$  and  $a \neq 3$

$q$ :  $a^2 + b^2 \neq c^2$

where  $n$  is an integer.

We use an indirect proof (proof by contrapositive).

*Hypothesis*:  $a^2 + b^2 = c^2$

Since  $a, b$  and  $c$  are consecutive:  $b = a + 1$  and  $c = a + 2$ . Replacing in the hypothesis, we get:

$$a^2 + b^2 = a^2 + a^2 + 2a + 1 = 2a^2 + 2a + 1$$

and

$$c^2 = (a + 2)^2 = a^2 + 4a + 4$$

Therefore:  $a^2 - 2a - 3 = 0$  or  $(a + 1)(a - 3) = 0$ . This quadratic equation has two solutions,  $a = -1$  or  $a = 3$ . Therefore  $\neg p$  is true. The original proposition is true.

### Exercise 3

Show that  $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, 2y = x + x^2$

Let  $x$  be an integer.  $x$  is either odd, or even. We check both cases:

Case 1:  $x$  is odd.

We show that  $x + x^2$  is even using a direct proof.  $x + x^2 = 2k + 1 + 2(2k^2 + 2k) + 1 = 2(2k^2 + 3k + 1)$  therefore  $x + x^2$  is even.

As  $x + x^2$  is even, there exists  $l \in \mathbb{Z}$  such that  $x + x^2 = 2l$ . We set  $y = l$ : we have shown the existence of  $y$  when  $x$  is odd.

Case 2:  $x$  is even.

We show that  $x + x^2$  is even. As  $x$  is even, there exists  $k \in \mathbb{Z}$  such that  $x = 2k$ . Then  $x + x^2 = 2k + 4k^2 = 2(k + 2k^2)$ . Therefore  $x + x^2$  is even.

As  $x + x^2$  is even, there exists  $l \in \mathbb{Z}$  such that  $x + x^2 = 2l$ . We set  $y = l$ : we have shown the existence of  $y$  when  $x$  is even.

We can conclude that there always exists  $y$  such that  $x + x^2 = 2y$ .

### Exercise 4

Show that for all  $n$  such that  $1 \leq n \leq 5$ ,  $n^2 - n + 11$  is prime.

We check systematically all integer values of  $n$  in  $[1, 5]$ :

- i)  $n = 1$  then  $n^2 - n + 11 = 11$ , which is prime.
- ii)  $n = 2$  then  $n^2 - n + 11 = 13$ , which is prime.
- iii)  $n = 3$  then  $n^2 - n + 11 = 17$ , which is prime.
- iv)  $n = 4$  then  $n^2 - n + 11 = 23$ , which is prime.
- v)  $n = 5$  then  $n^2 - n + 11 = 31$ , which is prime.

The proposition is therefore true.

### Exercise 5

Show that every odd integer can be written as the difference between two perfect squares.

Let  $n$  be an odd integer. There exists  $k \in \mathbb{Z}$  such that  $n = 2k + 1$ .

We notice that  $(k+1)^2 - k^2 = 2k + 1 = n$ . Therefore,  $n$  can be written as the difference between  $(k+1)^2$  and  $k^2$ , which are both perfect squares. The property is true.

## Extra credit

You arrive in a country called Transylvania whose inhabitants are humans and vampires. Humans always tell the truth, while vampires always lie. However, both humans and vampires can be sane or insane. If an inhabitant is insane, she will believe that a truth statement is false, and a false statement is true. Sane inhabitants believe that truth statements are true and false statements are false. Thus sane humans and insane vampires make only true statements, while insane humans and sane vampires make only false statements. You meet two inhabitants, A and B. You know that one of them is a human, and the other is a vampire. A tells you: we are both insane, while B tells you that at least one of us is sane. From this, can you find which one is the vampire?

Just like with all Smullyan's problems, we build a truth table. A and B can each be a human or a vampire, sane or insane. This would give 16 possibilities for the pair. However, we know that one is human, and the other one is a vampire: this reduces the table to 8 possibilities.

Let  $P_A$  be: "We are both insane".

Let  $P_B$  be: "At least one of us is sane"

Line #	A	B	$P_A$	$P_B$
1	Human, Sane	Vampire, Sane	False	True
2	Human, Sane	Vampire, Insane	False	True
3	Human, Insane	Vampire, Sane	False	True
4	Human, Insane	Vampire, Insane	True	False
5	Vampire, Sane	Human, Sane	False	True
6	Vampire, Sane	Human, Insane	False	True
7	Vampire, Insane	Human, Sane	False	True
8	Vampire, Insane	Human, Insane	True	False

We can eliminate:

Line 1: A would be a sane human that lies

Line 2: A would be a sane human that lies

Line 3: B would be a sane vampire that tells the truth

Line 4: A would be an insane human that tells the truth

Line 6: B would be an insane human that tells the truth

Line 7: A would be an insane vampire that lies

Both lines 5 and 8 are compatible with the premises. In both cases, B is a human and A is a vampire. We do not know however if they are sane or not.