ECS20 (Fall 2016)

October 19, 2016

Part I: logic

1) Using truth tables, establish for each of the two propositions below if it is a tautology, a contradiction or neither.

a) $[\neg(p \to \neg r)] \vee [\neg q \leftrightarrow r]$

Let us define $A = \neg(p \to \neg r)$ and $B = \neg q \leftrightarrow r$

p	q	r	$\neg r$	$p \to \neg r$	A	$\neg q$	r	B	$A \vee B$
T	T	T	F	F	T	F	T	F	T
T	T	F	T	T	F	F	F	T	T
T	F	T	F	F	T	T	T	T	T
T	F	F	T	T	F	T	F	F	F
F	T	T	F	T	F	F	T	F	F
F	T	F	T	T	F	F	F	T	T
F	F	T	F	T	F	T	T	T	T
F	F	F	T	T	F	T	F	F	F

The proposition $A \vee B$ is neither a tautology nor a contradiction.

b) $[\neg(p \to \neg q)] \wedge [(\neg r \vee q) \leftrightarrow \neg p]$

Let us define $A = \neg(p \to \neg q$ and $B = [(\neg r \vee q) \leftrightarrow \neg p]$

p	q	r	$\neg q$	$p \to \neg q$	A	$\neg r$	$\neg r \vee q$	$\neg p$	B	$A \wedge B$
T	T	T	F	F	T	F	T	F	F	F
T	T	F	F	F	T	T	T	F	F	F
T	F	T	T	T	F	F	F	F	T	F
T	F	F	T	T	F	T	T	F	F	F
F	T	T	F	T	F	F	T	T	T	F
F	T	F	F	T	F	T	T	T	T	F
F	F	T	T	T	F	F	F	T	F	F
F	F	F	T	T	F	T	T	T	T	F

The proposition $A \wedge B$ is a contradiction

2) Let us play a logical game. You find yourself in front of three rooms whose doors are closed. One of these rooms contains a Lady, another a Tiger and the third room in empty. There is one sign on each door; you are told that the sign on the door of the room containing the Lady is true, the sign on the door of the room with the Tiger is false, and the sign on the door of the empty room could be either true or false. Here are the signs:

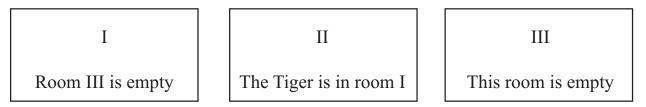| I	II	III
Room III is empty	The Tiger is in room I	This room is empty

Figure 1: The three rooms and their signs

We solve this problem using a table, just like for the knights and knaves problem. Let us define the symbol LA, TI and EM for the room containing the Lady, the Tiger and being Empty, respectively. We do not know in which order these rooms are, but we do know we have all three (i.e. we do not have 2 rooms containing a Tiger for example): there are 6 different ways to organize these three rooms. For each way, we analyze the three signs given and indicate if they are true (T) or false (F):

Line	Room I	Room II	Room III	Sign I	Sign II	sign III
1	LA	TI	EM	T	F	T
2	LA	EM	TI	F	F	F
3	TI	LA	EM	T	T	T
4	TI	EM	LA	F	T	F
5	EM	LA	TI	F	F	F
6	EM	TI	LA	F	F	F

Now we note that:

- line 1 is consistent.
- line 2 cannot be correct: one of the signs must be true as one room contains the Lady
- line 3 cannot be correct: one of the signs must be false as one room contains the Tiger
- line 4 cannot be correct: the sign on the Lady's room would be false.
- line 5 cannot be correct: one of the signs must be true as one room contains the Lady
- line 6 cannot be correct: one of the signs must be true as one room contains the Lady

Line 1 is the only valid possibility, therefore the Lady is in room I, the tiger in room II, and room III is empty.

Part II: proofs

Exercise 1

Let n be an integer. Show that n is even if and only if $n + n^2 - n^3$ is even

This is a proof of a biconditional. We do it in two steps:

a) We want to show that if n is even then $n + n^2 - n^3$ is even

This is an implication of the form $p \to q$, with:

p: n is even

q: $n + n^2 - n^3$ is even

where n is an integer.

Hypothesis: n is even: there exists k in $\mathbb{Z}$ such that $n = 2k$. Then $n + n^2 - n^3 = 2k + 4k^2 - 8k^3 = 2(k + 2k^2 - 4k^3)$, i.e. $n + n^2 - n^3$ is even. Therefore $p \to q$ is true.

b) We want to show that if $n + n^2 - n^3$ is even then n is even

This is an implication of the form $q \to p$, with:

q: $n + n^2 - n^3$ is even

p: n is even

where n is an integer. We use an indirect proof.

Hypothesis: $\neg p$ is true, i.e. n is odd. There exists k in $\mathbb{Z}$ such that $n = 2k + 1$. Then $n + n^2 - n^3 = 2k + 1 + (2k+1)^2 - (2k+1)^3 = 2k + 1 + 4k^2 + 4k + 1 - 8k^3 - 12k^2 - 6k - 1 = -8k^3 - 8k^2 + 1 = 2(-4k^3 - 4k^2) + 1$, which is odd. Therefore $\neg p \to \neg q$ is true, i.e. $q \to p$ is true.

We have shown $p \to q$ and $q \to p$ are true... therefore $p \leftrightarrow q$ is true.

Exercise 2

Let n be an integer. Show that $n^2 - n$ is even.

We use a proof by case:

i) Case 1: n is even. There exists an integer k such that $n = 2k$. Then $n^2 - n = 4k^2 - 2k = 2(2k^2 - k)$, i.e. $n^2 - n$ is even

ii) Case 2: n is odd. There exists an integer k such that $n = 2k + 1$. Then $n^2 - n = 4k^2 + 4k + 1 - 2k - 1 = 2(2k^2 + k)$, i.e. $n^2 - n$ is even

In all cases, we have shown that $n^2 - n$ is even.

Exercise 3

Let x be a real number. Show that if $x^3 + x^2 - 2x < 0$ then $x < 1$

This is an implication of the form $p \to q$, with:

p: $x^3 + x^2 - 2x < 0$

q: $x < 1$

where x is a real number.
We use an indirect proof (proof by contrapositive).
Hypothesis: $\neg q$: $x \geq 1$.
  We note that: $x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x - 1)(x + 2)$
  Note that since $x > 1$, $x > 0$, $x - 1 > 0$ and $x + 2 > 0$... therefore $x^3 + x^2 - 2x > 0$.
  We have shown $\neg q \to \neg p$ is true, therefore $p \to q$ is true.

Extra credit

Show that any rational number r can be written as the product of two irrational numbers.
  Let us write:

$$\begin{aligned} r &= ab \\ &= \sqrt{2}\frac{r}{\sqrt{2}} \end{aligned}$$

  We note that $a = \sqrt{2}$ is irrational. Let us show that $b = \frac{r}{\sqrt{2}}$ is irrational. We use a proof by contradiction.
  Assumption: b is rational, i.e. $\frac{r}{\sqrt{2}}$ is rational. Then,

$$r = \sqrt{2}b$$

Therefore $\sqrt{2} = \frac{r}{b}$. Since b and r are rational, we would conclude that $\sqrt{2}$ is rational.... but this is a contradiction.
  Therefore the assumption is wrong, and consequently, any rational number r can be written as the product of two irrational numbers.