# Midterm Review
# Solutions

ECS 20

Patrice Koehl
koehl@cs.ucdavis.edu

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## Exercise 1

Build a truth table for the proposition $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

| $p$ | $q$ | $\neg q$ | $p \leftrightarrow q$ | $(p \leftrightarrow \neg q)$ | $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$ |
|---|---|---|---|---|---|
| T | T | F | T | F | T |
| T | F | T | F | T | T |
| F | T | F | F | T | T |
| F | F | T | T | F | T |

Column 6 shows that $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$ is a tautology.

## Exercise 2

We design different proofs of the fact that the square of an even number is an even number. Let $p$ be the proposition "'$n$ is an even number"' and let $q$ be the proposition "'$n^2$ is an even number, where n is an integer.

(i) **Direct proof: $p \rightarrow q$.** To prove that an implication of the form $p \rightarrow q$ is true, it is sufficient to prove that if $p$ is true, then $q$ is true. Let us assume $p$ is true, i.e. $n$ is even. We know that there exists an (unique) integer $k$ such that $n = 2k$. By substitution, we get $n^2 = 4k^2 = 2(2k^2)$. This shows that $n^2$ is divisible by 2 and therefore even by definition. Hence $q$ is true, and the implication is always true.

(ii) **Indirect proof: $\neg q \rightarrow \neg p$.** In an indirect proof, we attempt to prove the contrapositive of the original implication (this is a valid proof technique, as we know that an implication and its contrapositive are equivalent). We suppose $\neg q$ is true, i.e. $n^2$ is odd, and we want to prove that $\neg p$ is true, i.e. $n$ is odd. We use our knowledge from number theory! $n^2$ is odd means that there exists a (unique) k such that $n^2 = 2k + 1$. Then $n^2 - 1 = 2k$. By definition, this means that 2 divides $(n-1)(n+1)$. Since 2 is prime, using Euclid's first proposition, we get that $2/(n-1)$ or $2/(n+1)$. If $2/(n-1)$, then there exists m such that $n - 1 = 2m$, hence

$n = 2m+1$ and n is odd, by definition. If $2/(n+1)$, then there exists m such that $n+1 = 2m$, hence $n = 2m - 1$, and n is odd, by definition. In all cases, n is odd, which concludes the proof.
An even simpler proof: since 2 is prime, according to Fermat's little theorem, $n^2 \equiv n(mod 2)$. Hence if $n^2 \equiv 1(mod 2)$, $n \equiv 1(mod 2)$.

(iii) **Proof by contradiction**. Given $p$ true, we assume that $\neg q$ is true, and we show that we reach a contradiction. Let $n$ be an even number, and let us assume that $n^2$ is an odd number. There exists k such that $n^2 = 2k + 1$. We show then (see indirect proof above) that $n$ is odd, which contradicts the premise (i.e. we have $p \wedge \neg p$, which is a contradiction). Hence the assumption $n^2$ is odd is false, and $n^2$ is even.

## Exercise 3

Suppose that $a$ is a non-zero rational number, and $b$ is an irrational number; we want to show that the product $ab$ is irrational. We use a proof by contradiction, i.e. we suppose that $ab$ is rational, and we attempt to show that this leads to a contradiction. Let us write $ab = c$, with c rational. Since $a$ is a non-zero rational, it has a multiplicative inverse, $a^{-1}$ that is also rational. Then $b = ca^{-1}$. Since the product of two rational numbers is rational, this shows that $b$ is rational which contradict the premise that $b$ is irrational. Hence the hypothesis $ab$ is rational is false, and $ab$ is therefore irrational.

## Exercise 4

Since there is an order relation on real numbers, given 2 real numbers, $x$ and $y$, there can be 3 cases, $x > y$, $x < y$, and $x = y$ (this is sometimes referred to as the trichotomy law).

a) When $x > y$, $max(x, y) = x$ and $min(x, y) = y$. In this case, $max(x, y) + min(x, y) = x + y$.

b) When $x = y$, $max(x, y) = min(x, y) = x = y$. In this case, $max(x, y) + min(x, y) = x + x = x + y$.

c) When $x < y$, $max(x, y) = y$ and $min(x, y) = x$. In this case, $max(x, y) + min(x, y) = y + x = x + y$, by commutative property of addition of real numbers.

The method of proof by cases allows us to conclude that $max(x, y) + min(x, y) = x + y$ for all $(x, y) \in \mathbb{R}^2$.

## Exercise 5

Let a= $65^{1000} - 8^{2001} + 3^{177}$, b= $79^{1212} - 9^{2399} + 2^{2001}$ and c= $24^{4493} - 5^{8192} + 7^{1777}$; we want to show that the product of two of these 3 numbers is non negative. In other words, we want to show that **ONE** of the elements of the set $\{ab, ac, bc\}$ is non negative. We develop a proof by contradiction. We suppose that **ALL** the elements of the set $\{ab, ac, bc\}$ are strictly negative. Let $P$ by the product of all the elements of that set. Since there are 3 negative elements in that set, $P$ is strictly negative. But $P = abacbc = a^2b^2c^2$, i.e. $P$ is the product of 3 positive numbers (three squares), hence $P$ is positive. We have shown that $P$ is both strictly negative and positive, i.e we have reached a contradiction. The hypothesis was wrong, and we therefore validate that the product of two of the 3 numbers a, b and c is non negative.

# Exercise 6

a) $x \in A \cup B \Rightarrow x \in A \vee x \in B$. Since $A \cup B \cup C$ contains all elements either in A, B or C, all the elements of $A \cup B$ are contained in $A \cup B \cup C$. Hence, proved that $A \cup B \subset A \cup B \cup C$.

b) We know that the conjunction logic operation is both associative and commutative. Checking membership of $(A - B) - C$:

$$\begin{aligned}
((A-B)-C) &= \{x \mid x \in ((A-B)-C)\} \\
&= \{x \mid x \in (A-B) \wedge \neg(x \in C)\} \\
&= \{x \mid (x \in A \wedge (\neg x \in B)) \wedge \neg(x \in C)\} \\
&= \{x \mid (\neg(x \in B) \wedge x \in A) \wedge \neg(x \in C)\} \\
&= \{x \mid \neg(x \in B) \wedge (x \in A \wedge \neg(x \in C)\} \\
&= \{x \mid \neg(x \in B) \wedge (x \in (A-C))\}
\end{aligned}$$

Thus, all elements of $(A-B)-C$ are contained in $(A-C)$ and not contained in $B$, which means that all elements of $(A-B)-C$ are elements of $(A-C)$. This proves that $(A-B)-C \subset (A-C)$.

c) Let us write the definition of $(B - A) \cup (C - A)$, and use logic operations:

$$\begin{aligned}
(B-A) \cup (C-A) &= \{x \mid x \in (B-A) \vee x \in (C-A)\} \\
&= \{x \mid (x \in B \wedge \neg(x \in A)) \vee (x \in C \wedge \neg(x \in A))\}
\end{aligned}$$

Since $\wedge$ is commutative, we obtain:

$$(B-A) \cup (C-A) = \{x \mid (\neg(x \in A) \wedge x \in B) \vee (\neg(x \in A) \wedge x \in C)\}$$

Since $\wedge$ and $\vee$ are associative, we obtain:

$$\begin{aligned}
(B-A) \cup (C-A) &= \{x \mid (\neg(x \in A)) \wedge (x \in B \vee x \in C)\} \\
&= \{x \mid (\neg(x \in A) \wedge (x \in B \cup C)\} \\
&= \{x \mid x \in ((B \cup C) - A)\}
\end{aligned}$$

This completes the proof that $(B - A) \cup (C - A) = (B \cup C) - A$.