Midterm 2: Solutions

ECS20 (Spring 2016)

May 18, 2016

Part I: logic

Exercise 1: Smullyan's island

A very special island is inhabited only by Knights and Knaves. Knights always tell the truth, while Knaves always lie. You meet two inhabitants: Sally and Claire. You know that one of them is the Queen of the island. Sally says, "Claire is the Queen and she is a Knave". Claire says, "Sally is not the Queen but she is a Knight". Can you find out if Sally is a Knight or Knave? Can you find out if Claire is a Knight or Knave? Can you tell me who is the Queen? Explain your answer.

Let us build the table for the possible options for Sally and Claire (both can be a Knight of a Knave, a Queen or not; only one can be the Queen). We then check the validity of the two statements, and finally check the consistency of the truth values for those statements with the nature of Sally and Claire.

Line	Sally	Claire	Sally says	Claire says
1	Knight, Queen	Knight, not Queen	F	F
2	Knight, Queen	Knave, not Queen	F	F
3	Knight, not Queen	Knight, Queen	F	T
4	Knight, not Queen	Knave, Queen	T	T
5	Knave, Queen	Knight, not Queen	F	F
6	Knave, Queen,	Knave, not Queen	F	F
7	Knave, not Queen	Knight, Queen	F	F
8	Knave, not Queen	Knave, Queen	T	F

Compatibility:

Line 1: No: Sally would be a Knight who lies

Line 2: No, Sally would be a Knight who lies

Line 3: No, Sally would be a Knight who lies

Line 4: No, Claire would be a Knave who tells the truth

Line 5: No, Claire would be a Knight who lies

Line 6: Yes

Line 7: No, Claire would be a Knight who lies

Line 8: No, Sally would be a Knave who tells the truth

Therefore Sally and Claire are Knaves, and Sally is the Queen.

Part II: proofs and number theory

Exercise 1

Give a direct proof, an indirect proof and a proof by contradiction of the proposition: if n^3+1 is odd, then n is even, where n is a natural number.

This is a problem of showing a conditional $p \to q$ is true, where
$p : n^3+1$ is odd $q : n$ is even

We will use three different types of proof: direct, indirect, and proof by contradiction

a) Direct proof: we show directly that $p \to q$ is true.

   Hypothesis: p is true, n^3+1 is odd. Therefore there exists an integer k such that $n^3+1 = 2k+1$, i.e. $n^3 = 2k$. Therefore 2 divides n^3. Since 2 is a prime number, according to Euclid's theorem, we conclude that 2 divides n, therefore n is even. We have showed that q is true, therefore $p \to q$ is true

b) Indirect proof: we show that $\neg q \to \neg p$ is true.

   Hypothesis: $\neg q$ is true, therefore n is odd. There exists an integer k such that $n = 2k+1$. Therefore,

   $n^3+1 = (2k+1)^3+1 = 8k^3+12k^2+6k+1+1 = 2(4k^3+6k^2+3k+1)$

   Since $4k^3+6k^2+3k+1$ is integer, n^3+1 is even.

   We have shown that $\neg q \to \neg p$ is true, therefore $p \to q$ is true

c) Proof by contradiction: we suppose $p \to q$ is false

   Hypothesis: $p \to q$ is false, i.e. p is true and $\neg q$ is true, namely n^3+1 is odd and n is odd.

   Since n is odd, there exists an integer k such that $n = 2k+1$. Therefore, $n^3+1 = (2k+1)^3+1 = 8k^3+12k^2+6k+1+1 = 2(4k^3+6k^2+3k+1)$

   Since $4k^3+6k^2+3k+1$ is integer, n^3+1 is even. But we have supposed that n^3+1 is odd. We have reached a contradiction. Therefore the hypothesis we made is false, therefore $p \to q$ is true.

Exercise 2

Show that for all integers $n > 1, n^3+3n^2+2n$ is divisible by 2 and 3. (Hint: one possibility is to use Fermat's little theorem)

Let n be a positive integer strictly greater than 1. Let us use the hint given to us.

a) Divisibility by 2.

   Since 2 is prime, for all $n > 1$, $n^2 \equiv n[2]$. Multiplying the congruence by n, we get $n^3 \equiv n^2 \equiv n[2]$.

Therefore,

$$\begin{aligned} n^3+3n^2+2n &\equiv n+3n+2n[2] \\ &\equiv 6n[2] \\ &\equiv 0[2] \end{aligned}$$

Therefore n^3+3n^2+2n is divisible by 2.

a) Divisibility by 3.

Since 3 is prime, for all $n>1$, $n^3 \equiv n[3]$.

Therefore,

$$\begin{aligned} n^3+3n^2+2n &\equiv n+3n^2+2n[3] \\ &\equiv n+2n[3] \\ &\equiv 3n[3] \\ &\equiv 0[3] \end{aligned}$$

Therefore n^3+3n^2+2n is divisible by 3.

Exercise 3

Show that the sum of any three consecutive perfect cubes is divisible by 9 (Note: a perfect cube is a number that can be written in the form n^3 where n is an integer. The three numbers $(n-1)^3$, n^3 and $(n+1)^3$are three consecutive perfect cubes. Hint: Start by showing that $n^3+2n \equiv 0[3]$, for all integer n).

Let us follow the hint. We show first that n^3+2n is a multiple of 3, which is equivalent to $n^3+2n \equiv 0[3]$, for all integer n. Let n be an integer. Since 3 is a prime number, we can use Fermat's little theorem:

$$n^3 \equiv n[3]$$

Therefore

$$\begin{aligned} n^3+2n &\equiv n+2n[3] \\ &\equiv 3n[3] \\ &\equiv 0[3] \end{aligned}$$

Therefore $n^3+2n \equiv 0[3]$, for all integer n, namely n^3+2n is a multiple of 3.

Let us consider now three consecutive perfect squares: $(n-1)^3$, n^3 and $(n+1)^3$. Their sum satisfies

$$\begin{aligned} S &= (n-1)^3+n^3+(n+1)^3 \\ &= n^3-3n^2+3n-1+n^3+n^3+3n^2+3n+1 \\ &= 3n^3+6n \\ &= 3(n^3+2n) \end{aligned}$$

We have showed that $n^3 + 2n$ is a multiple of 3: there exits k integer such that $n^3 + 2n = 3k$. Therefore,

$$\begin{aligned} S &= 3(n^3 + 2n) \\ &= 9k \end{aligned}$$

Therefore S is a multiple of 9.

Exercise 4

Evaluate the remainder of the division of 2^{473} by 13.

Let A $= 2^{473}$. Note first that $473 = 13 \times 36 + 5$, therefore $A = (2^{36})^{13} \times 2^5$.
Since 13 is prime, we can use Fermat's little theorem, i.e. for all natural number a,

$$a^{13} \equiv a[13]$$

Therefore,

$$\begin{aligned} A &\equiv (2^{36})^{13} \times 2^5[13] \\ &\equiv 2^{36} \times 2^5[13] \\ &\equiv 2^{41}[13] \end{aligned}$$

Now we note that $41 = 3 \times 13 + 2$, therefore $2^{41} = (2^3)^{13} \times 2^2$. Therefore:

$$\begin{aligned} A &\equiv 2^{41}[13] \\ &\equiv (2^3)^{13} \times 2^2[13] \\ &\equiv 2^3 \times 2^2[13] \\ &\equiv 2^5[13] \end{aligned}$$

Since $2^5 = 32$, and $32 = 2 \times 13 + 6$, $2^5 \equiv 6[13]$.
Therefore

$$A \equiv 6[13]$$

and the remainder of the division of 2^{473} by 13 is 6.

Part III: sets and functions

Exercise 1

Let A and B be two sets in a domain D. Show that $(\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) \cup (A \cap B) = \overline{A} \cup B$.

Let $LHS = (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) \cup (A \cap B)$ and $RHS = \overline{A} \cup B$.
Using the distributivity of $\cap$ and $\cup$, we have

$$\begin{aligned} (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) &= \overline{A} \cap (B \cup \overline{B}) \\ &= \overline{A} \cap D \\ &= \overline{A} \end{aligned}$$

Therefore

$$\begin{aligned} LHS &= \overline{A} \cup (A \cap B) \\ &= (\overline{A} \cup A) \cap (\overline{A} \cup B \\ &= D \cap (\overline{A} \cup B \\ &= (\overline{A} \cup B \\ &= RHS \end{aligned}$$

Therefore the property is true.

Exercise 2

[There was a typo in the midterm: it was written that a and b are real numbers, while they should be integers... the grading was designed so that no one was penalized by this typo.]

Let a, and b be two strictly positive integers and let x be a real number.. Show that:

$$\left\lfloor \frac{\lfloor \frac{x}{a} \rfloor}{b} \right\rfloor = \left\lfloor \frac{x}{ab} \right\rfloor$$

Let us define $k = \lfloor \frac{x}{a} \rfloor$ and $m = \lfloor \frac{x}{ab} \rfloor$. By definition of floor, we have the two properties:
$k \leq \frac{x}{a} < k + 1$
and
$m \leq \frac{x}{ab} < m + 1$
Let us multiply the second inequalities by b:
$bm \leq \frac{x}{a} < b(m + 1)$
We notice that:
$k \leq \frac{x}{a}$ and $\frac{x}{a} < b(m + 1)$; therefore $k < b(m + 1)$.
$k \leq \frac{x}{a}$ and $bm \leq \frac{x}{a}$. Therefore k and bm are two integers smaller than $\frac{x}{a}$. By definition of floor, k is the largest integer smaller that $\frac{x}{a}$. Therefore $bm \leq k$.

Combining those two inequalities, we get $bm \leq k < b(m+1)$. After division by b, $m < \frac{k}{b} < m+1$. Therefore m is the floor of $\frac{k}{b}$. Replacing m and k by their values, we get:

$$m = \left\lfloor \frac{x}{ab} \right\rfloor = \left\lfloor \frac{k}{b} \right\rfloor = \left\lfloor \frac{\lfloor \frac{x}{a} \rfloor}{b} \right\rfloor$$

The property is therefore true.

Extra credit

Let x be a positive real number. Solve $\lfloor x \lfloor x \rfloor \rfloor = 5$.

Let $A = \lfloor x \lfloor x \rfloor \rfloor$.
Since $x \geq 0$, we do not need to worry about x being negative.
We notice first that if $x \geq 3$, then $\lfloor x \rfloor \geq 3$, and $x \lfloor x \rfloor \geq 9$, therefore $A \geq 9$.
Therefore possible solutions for x are between 0 and 3, 3 not included. We look at three cases:

a) $0 \leq x < 1$

In this case, $\lfloor x \rfloor = 0$ and $A = 0$. There are no solutions in this interval.

b) $1 \leq x < 2$

In this case, $\lfloor x \rfloor = 1$ and $A = \lfloor x \rfloor = 1$. There are no solutions in this interval.

c) $2 \leq x < 3$

In this case, $\lfloor x \rfloor = 2$ and $A = \lfloor 2x \rfloor$. Since $2 \leq x < 3$, $4 \leq 2x < 6$. We distinguish two cases:

  i) $4 \leq 2x < 5$, namely $2 \leq x < 2.5$. Then $A = \lfloor 2x \rfloor = 4$; there are no solutions in this interval.

  ii) $5 \leq 2x < 6$, namely $2.5 \leq x < 3$. Then $A = \lfloor 2x \rfloor = 5$; all values of x in this interval are solutions.

In conclusion, all values of $x \in [2.5, 3[$ are solutions of the equation.