Midterm 2: Solutions

ECS20 (Fall 2016)

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Part I: Sets

Let A and B be two sets in a domain D. Show that $\overline{(A\bigcap\overline{B})\bigcup(B\bigcap\overline{A})} = (\overline{A}\bigcap\overline{B})\bigcup(B\bigcap A)$.

We can use a proof by membership table. I ill use the set identities. Let $LHS = \overline{(A\bigcap\overline{B})\bigcup(B\bigcap\overline{A})}$ and $RHS = (\overline{A}\bigcap\overline{B})\bigcup(B\bigcap A)$.

Then:

$$\begin{aligned}
LHS &= \overline{(A\bigcap\overline{B})}\bigcap\overline{(B\bigcap\overline{A})} \\
&= (\overline{A}\bigcup B)\bigcap(\overline{B}\bigcup A) \\
&= \left[(\overline{A}\bigcup B)\bigcap\overline{B}\right]\bigcup\left[(\overline{A}\bigcup B)\bigcap A\right] \\
&= \left[\overline{B}\bigcap(\overline{A}\bigcup B)\right]\bigcup\left[A\bigcap(\overline{A}\bigcup B)\right] \\
&= \left[(\overline{B}\bigcap\overline{A})\bigcup(B\bigcap\overline{B})\right]\bigcup\left[(A\bigcap\overline{A})\bigcup(A\bigcap B)\right] \\
&= \left[(\overline{B}\bigcap\overline{A})\bigcup\emptyset\right]\bigcup\left[\emptyset\bigcup(A\bigcap B)\right] \\
&= (\overline{B}\bigcap\overline{A})\bigcup(A\bigcap B) \\
&= RHS
\end{aligned}$$

Therefore the two sets LHR and RHS are equal!

Part II: functions

1) Let x be a real number. Solve $\lfloor 3x-2\rfloor = x$.

   We notice first that since floor is a function from $\mathbb{R}$ to $\mathbb{Z}$, x has to be an integer. Since x is an integer, $3x-2$ is an integer. Therefore the equation becomes $3x-2=x$; this leads to $x=1$.

2) Let x be a real number. Show that $\lfloor \frac{x}{2}\rfloor + \lfloor \frac{x+1}{2}\rfloor = \lfloor x\rfloor$

   Let $\lfloor x\rfloor = n$, where n is an integer. By definition of floor, we have:

   $n \leq x < n+1$.

   We consider two cases:

   1) n **is even:** there exists an integer k such that $n=2k$. We can rewrite the inequality above as:

      $$2k \leq x < 2k+1$$

      Then

      $$k \leq \frac{x}{2} < k + \frac{1}{2} < k+1$$

Therefore

$$\lfloor \frac{x}{2} \rfloor = k. \tag{1}$$

Similarly,

$$2k + 1 \leq x + 1 < 2k + 2$$

Then

$$k < k + \frac{1}{2} \leq \frac{x+1}{2} < k + 1$$

Therefore

$$\lfloor \frac{x+1}{2} \rfloor = k \tag{2}$$

Combining equations (1) and (2), we get $\lfloor \frac{x}{2} \rfloor + \lfloor \frac{x+1}{2} \rfloor = 2k = n = \lfloor x \rfloor$

1) n **is odd:** there exists an integer k such that $n = 2k + 1$. We can rewrite the inequality above as:

$$2k + 1 \leq x < 2k + 2$$

Then

$$k < k + \frac{1}{2} < \frac{x}{2} < k + 1$$

Therefore

$$\lfloor \frac{x}{2} \rfloor = k. \tag{3}$$

Similarly,

$$2k + 2 \leq x + 1 < 2k + 3$$

Then

$$k + 1 \leq \frac{x+1}{2} < k + \frac{3}{2} < k + 2$$

Therefore

$$\lfloor \frac{x+1}{2} \rfloor = k + 1 \tag{4}$$

Combining equations (3) and (4), we get $\lfloor \frac{x}{2} \rfloor + \lfloor \frac{x+1}{2} \rfloor = k + k + 1 = n = \lfloor x \rfloor$

Part III: Number theory

1) Let a, b, and c be three natural numbers. Show that if b/a, c/a and $\gcd(b,c)=1$, then $(bc)/a$.

   We do a direct proof. Our hypothesis is that b/a, c/a and $\gcd(b,c)=1$. From the last property, based on Bezout's identity, we know that there exits two integer numbers k and l such that:

   $$kb + lc = 1$$

   After mulitplication by a,

   $$kba + lca = a$$

   We know that b/a. There exists an integer n such that $a = bn$. Similarly, we know that c/a. Therefore, there exists an integer m such that $a = cm$. Replacing in the equation above, we get:

   $$kbcm + lcbn = a$$

   After factorizing bc, we get:

   $$bc(km + ln) = a$$

   Therefore $(bc)/a$.

2) Show that there are no integer solutions to the equation $x^2 - 3y^2 = -1$.

   We do a proof by contradiction. Let us suppose that there exists a pair of integers (x_0, y_0) such that $x_0^2 - 3y_0^2 = -1$. Let us define $LHS = x_0^2 - 3y_0^2$ and $RHS = -1$. We take those two numbers modulo 3:

   $RHS \equiv -1 \quad [3]$ therefore $RHS \equiv 2 \quad [3]$.

   $LHS \equiv x_0^2 \quad [3]$ since $3y_0^2$ is a multiple of 3. Let us consider the division of x_0 by 3:

   There exists an integer k and an integer r such that $x_0 = 3k + r$, with $r \in 0, 1, 2$. Then:

   $$x_0^2 = 9k^2 + 6k + r^2$$

   Therefore $x_0^2 \equiv r^2 \quad [3]$. Since $r \in 0, 1, 2$, $r^2 \in 0, 1, 4$. This means that the remainder of the division of x_0^2 by 3 is either 0 or 1, and therefore $LHS \equiv 0 \quad [3]$ or $LHS \equiv 1 \quad [3]$. This however contradicts that $LHS = RHS$.

   As we have reached a contradiction, there are no integer solutions to the equation $x^2 - 3y^2 = -1$.

3) Show that 13 divides $3^{126} + 5^{126}$.

   Let us define $A = 3^{126}$ and $B = 5^{126}$.

   We notice first that 13 is a prime number. We have $126 = 13 \times 9 + 9$. Therefore:

   $$A = (3^9)^1 3 \times 3^9$$

Applying Fermat's little theorem, we get:

$$\begin{aligned} A &\equiv 3^9 \times 3^9 \quad [13] \\ &\equiv 3^{18} \quad [13] \\ &\equiv 3^{13} \times 3^5 \quad [13] \\ &\equiv 3^6[\quad 13] \end{aligned}$$

Notice that $3^3 \equiv 1 \quad [13]$. We have $3^6 \equiv 1 \quad [13]$ and therefore $A \equiv 1 \quad [13]$.

Similarly,

$$B = (5^9)^1 3 \times 5^9$$

Applying Fermat's little theorem, we get:

$$\begin{aligned} B &\equiv 5^9 \times 5^9 \quad [13] \\ &\equiv 5^{18} \quad [13] \\ &\equiv 5^{13} \times 5^5 \quad [13] \\ &\equiv 5^6 \quad [13] \end{aligned}$$

Notice that $5^2 \equiv -1 \quad [13]$. We have $5^4 \equiv 1 \quad [13]$ and therefore $B \equiv -1 \quad [13]$, i.e. $B \equiv 12 \quad [13]$.

Then, $A + B \equiv 1 + 12 \quad [13]$, and therefore $A + B \equiv 0 \quad [13]$, i.e. 13 divides $3^{126} + 5^{126}$.

Extra credit

Let x be a real number. Find all positive (non-zero) solutions of $x\lfloor x \rfloor = x^2 - \lfloor x \rfloor^2$.

Let $\lfloor x \rfloor = n$, where n is an integer, and let $x = n + \epsilon$, where ϵ is a real number with $0 \leq \epsilon < 1$. Replacing in the equation, we get:

$$\begin{aligned} (n + \epsilon)n &= (n + \epsilon)^2 - n^2 \\ &= 2\epsilon n + \epsilon^2 \end{aligned}$$

Therefore

$$n^2 - \epsilon n = \epsilon^2 \tag{5}$$

Since $\epsilon < 1$ and n is positive (since we are looking for x is positive), $n\epsilon < n$, therefore $-n\epsilon > -n$ and $n^2 - n\epsilon > n^2 - n$. When $n \geq 2$, $n^2 - n \geq 2$, and therefore $n^2 - n\epsilon > 2$. Since $n^2 - \epsilon n = \epsilon^2$, this would lead to $\epsilon^2 > 2$, which is not possible since $\epsilon < 1$.

Therefore $n \leq 1$, and since x is positive, $n = 0$ or $n = 1$.

If $n = 0$, the equation become $0 = x$, but we are only looking at the non-zero solutions. Therefore $n = 1$.

Replacing in Equation (5), we get:

$$\epsilon^2 + \epsilon - 1 = 0$$

This equation has two solutions:

$$\epsilon_1 = \frac{-1+\sqrt{5}}{2}$$
$$\epsilon_2 = \frac{-1-\sqrt{5}}{2}$$

Only one of these two solutions is positive, ϵ_1. Therefore, there is only one non-zero positive solution to the equation,

$$x = n + \epsilon_1 = 1 + \frac{-1+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$$