erties divide the plane into $k^2 - k + 2 + 2k = (k^2 + 2k + 1) - (k + 1) + 2 = (k + 1)^2 - (k + 1) + 2$ regions. **51.** Suppose $\sqrt{2}$ were rational. Then $\sqrt{2} = a/b$, where a and b are positive integers. It follows that the set $S = \{n\sqrt{2} \mid n \in \mathbf{N}\} \cap \mathbf{N}$ is a nonempty set of positive integers, because $b\sqrt{2} = a$ belongs to S. Let t be the least element of S, which exists by the well-ordering property. Then $t = s\sqrt{2}$ for some integer s. We have $t - s = s\sqrt{2} - s = s(\sqrt{2} - 1)$, so $t - s$ is a positive integer because $\sqrt{2} > 1$. Hence, $t - s$ belongs to S. This is a contradiction because $t - s = s\sqrt{2} - s < s$. Hence, $\sqrt{2}$ is irrational. **53. a)** Let $d = \gcd(a_1, a_2, \ldots, a_n)$. Then d is a divisor of each a_i and so must be a divisor of $\gcd(a_{n-1}, a_n)$. Hence, d is a common divisor of $a_1, a_2, \ldots, a_{n-2}$, and $\gcd(a_{n-1}, a_n)$. To show that it is the greatest common divisor of these numbers, suppose that c is a common divisor of them. Then c is a divisor of a_i for $i = 1, 2, \ldots, n-2$ and a divisor of $\gcd(a_{n-1}, a_n)$, so it is a divisor of a_{n-1} and a_n. Hence, c is a common divisor of $a_1, a_2, \ldots, a_{n-1}$, and a_n. Hence, it is a divisor of d, the greatest common divisor of $a_1, a_2, \ldots, a_n$. It follows that d is the greatest common divisor, as claimed. **b)** If $n = 2$, apply the Euclidean algorithm. Otherwise, apply the Euclidean algorithm to a_{n-1} and a_n, obtaining $d = \gcd(a_{n-1}, a_n)$, and then apply the algorithm recursively to $a_1, a_2, \ldots, a_{n-2}, d$. **55.** $f(n) = n^2$. Let $P(n)$ be "$f(n) = n^2$." *Basis step:* $P(1)$ is true because $f(1) = 1 = 1^2$, which follows from the definition of f. *Inductive step:* Assume $f(n) = n^2$. Then $f(n+1) = f((n+1) - 1) + 2(n+1) - 1 = f(n) + 2n + 1 = n^2 + 2n + 1 = (n+1)^2$. **57. a)** λ, 0, 1, 00, 01, 11, 000, 001, 011, 111, 0000, 0001, 0011, 0111, 1111, 00000, 00001, 00011, 00111, 01111, 11111 **b)** $S = \{\alpha\beta \mid \alpha$ is a string of m 0s and β is a string of n 1s, $m \geq 0, n \geq 0\}$ **59.** Apply the first recursive step to λ to get () ∈ B. Apply the second recursive step to this string to get ()() ∈ B. Apply the first recursive step to this string to get (()()) ∈ B. By Exercise 62, (())) is not in B because the number of left parentheses does not equal the number of right parentheses. **61.** λ, (), (()), ()() **63. a)** 0 **b)** −2 **c)** 2 **d)** 0

65.

procedure *generate*(n: nonnegative integer)
if n is odd **then**
  $S := S(n-1)$ {the S constructed by *generate*$(n-1)$}
  $T := T(n-1)$ {the T constructed by *generate*$(n-1)$}
else if $n = 0$ **then**
  $S := \emptyset$
  $T := \{\lambda\}$
else
  $S' := S(n-2)$ {the S constructed by *generate*$(n-2)$}
  $T' := T(n-2)$ {the T constructed by *generate*$(n-2)$}
  $T := T' \cup \{(x) \mid x \in T' \cup S' \wedge \text{length}(x) = n-2\}$
  $S := S' \cup \{xy \mid x \in T' \wedge y \in T' \cup S' \wedge \text{length}(xy) = n\}$
{$T \cup S$ is the set of balanced strings of length at most n}

67. If $x \leq y$ initially, then $x := y$ is not executed, so $x \leq y$ is a true final assertion. If $x > y$ initially, then $x := y$ is executed, so $x \leq y$ is again a true final assertion.

69. procedure *zerocount*$(a_1, a_2, \ldots, a_n$: list of integers)
 if $n = 1$ **then**
  if $a_1 = 0$ **then return** 1
  else return 0
 else
  if $a_n = 0$ **then return** *zerocount*$(a_1, a_2, \ldots, a_{n-1}) + 1$
  else return *zerocount*$(a_1, a_2, \ldots, a_{n-1})$

71. We will prove that $a(n)$ is a natural number and $a(n) \leq n$. This is true for the base case $n = 0$ because $a(0) = 0$. Now assume that $a(n-1)$ is a natural number and $a(n-1) \leq n-1$. Then $a(a(n-1))$ is a applied to a natural number less than or equal to $n-1$. Hence, $a(a(n-1))$ is also a natural number minus than or equal to $n-1$. Therefore, $n - a(a(n-1))$ is n minus some natural number less than or equal to $n-1$, which is a natural number less than or equal to n. **73.** From Exercise 72, $a(n) = \lfloor (n+1)\mu \rfloor$ and $a(n-1) = \lfloor n\mu \rfloor$. Because $\mu < 1$, these two values are equal or they differ by 1. First suppose that $\mu n - \lfloor \mu n \rfloor < 1 - \mu$. This is equivalent to $\mu(n+1) < 1 + \lfloor \mu n \rfloor$. If this is true, then $\lfloor \mu(n+1) \rfloor = \lfloor \mu n \rfloor$. On the other hand, if $\mu n - \lfloor \mu n \rfloor \geq 1 - \mu$, then $\mu(n+1) \geq 1 + \lfloor \mu n \rfloor$, so $\lfloor \mu(n+1) \rfloor = \lfloor \mu n \rfloor + 1$, as desired. **75.** $f(0) = 1$, $m(0) = 0$; $f(1) = 1$, $m(1) = 0$; $f(2) = 2$, $m(2) = 1$; $f(3) = 2$, $m(3) = 2$; $f(4) = 3$, $m(4) = 2$; $f(5) = 3$, $m(5) = 3$; $f(6) = 4$, $m(6) = 4$; $f(7) = 5$, $m(7) = 4$; $f(8) = 5$, $m(8) = 5$; $f(9) = 6$, $m(9) = 6$ **77.** The last occurrence of n is in the position for which the total number of 1s, 2s, ..., ns all together is that position number. But because a_k is the number of occurrences of k, this is just $\sum_{k=1}^{n} a_k$, as desired. Because $f(n)$ is the sum of the first n terms of the sequence, $f(f(n))$ is the sum of the first $f(n)$ terms of the sequence. But because $f(n)$ is the last term whose value is n, this means that the sum is the sum of all terms of the sequence whose value is at most n. Because there are a_k terms of the sequence whose value is k, this sum is $\sum_{k=1}^{n} k \cdot a_k$, as desired

CHAPTER 6

Section 6.1

1. a) 5850 **b)** 343 **3. a)** 4^{10} **b)** 5^{10} **5.** 42 **7.** 26^3 **9.** 676 **11.** 2^8 **13.** $n + 1$ (counting the empty string) **15.** 475,255 (counting the empty string) **17.** 1,321,368,961 **19. a)** 729 **b)** 256 **c)** 1024 **d)** 64 **21. a)** Seven: 56, 63, 70, 77, 84, 91, 98 **b)** Five: 55, 66, 77, 88, 99 **c)** One: 77 **23. a)** 128 **b)** 450 **c)** 9 **d)** 675 **e)** 450 **f)** 450 **g)** 225 **h)** 75 **25. a)** 990 **b)** 500 **c)** 27 **27.** 3^{50} **29.** 52,457,600 **31.** 20,077,200 **33. a)** 37,822,859,361 **b)** 8,204,716,800 **c)** 40,159,050, 880 **d)** 12,113,640,000 **e)** 171,004,205,215 **f)** 72,043,541,640 **g)** 6,230,721,635 **h)** 223,149,655 **35. a)** 0 **b)** 120 **c)** 720 **d)** 2520 **37. a)** 2 if $n = 1$, 2 if $n = 2$, 0 if $n \geq 3$ **b)** 2^{n-2} for $n > 1$; 1 if $n = 1$ **c)** $2(n-1)$ **39.** $(n+1)^m$ **41.** If n is even, $2^{n/2}$; if n is odd, $2^{(n+1)/2}$ **43. a)** 175 **b)** 248 **c)** 232 **d)** 84 **45.** 60 **47. a)** 240 **b)** 480 **c)** 360 **49.** 352 **51.** 147 **53.** 33 **55. a)** 9,920,671,339,261,325,541,376 $\approx 9.9 \times 10^{21}$ **b)** 6,641,514,961,387,068,437,760 $\approx 6.6 \times 10^{21}$ **c)** About 314,000 years **57.** $54(64^{65536} - 1)/63$

59. 7,104,000,000,000 **61.** $16^{10} + 16^{26} + 16^{58}$ **63.** 666,667 **65.** 18 **67.** 17 **69.** 22 **71.** Let $P(m)$ be the sum rule for m tasks. For the basis case take $m = 2$. This is just the sum rule for two tasks. Now assume that $P(m)$ is true. Consider $m + 1$ tasks, $T_1, T_2, \ldots, T_m, T_{m+1}$, which can be done in $n_1, n_2, \ldots, n_m, n_{m+1}$ ways, respectively, such that no two of these tasks can be done at the same time. To do one of these tasks, we can either do one of the first m of these or do task T_{m+1}. By the sum rule for two tasks, the number of ways to do this is the sum of the number of ways to do one of the first m tasks, plus n_{m+1}. By the inductive hypothesis, this is $n_1 + n_2 + \cdots + n_m + n_{m+1}$, as desired. **73.** $n(n - 3)/2$

Section 6.2

1. Because there are six classes, but only five weekdays, the pigeonhole principle shows that at least two classes must be held on the same day. **3. a)** 3 **b)** 14 **5.** Because there are four possible remainders when an integer is divided by 4, the pigeonhole principle implies that given five integers, at least two have the same remainder. **7.** Let $a, a + 1, \ldots, a + n - 1$ be the integers in the sequence. The integers $(a + i)$ **mod** n, $i = 0, 1, 2, \ldots, n - 1$, are distinct, because $0 < (a + j) - (a + k) < n$ whenever $0 \le k < j \le n - 1$. Because there are n possible values for $(a + i)$ **mod** n and there are n different integers in the set, each of these values is taken on exactly once. It follows that there is exactly one integer in the sequence that is divisible by n. **9.** 4951 **11.** The midpoint of the segment joining the points (a, b, c) and (d, e, f) is $((a+d)/2, (b+e)/2, (c+f)/2)$. It has integer coefficients if and only if a and d have the same parity, b and e have the same parity, and c and f have the same parity. Because there are eight possible triples of parity [such as (*even*, *odd*, *even*)], by the pigeonhole principle at least two of the nine points have the same triple of parities. The midpoint of the segment joining two such points has integer coefficients. **13. a)** Group the first eight positive integers into four subsets of two integers each so that the integers of each subset add up to 9: {1, 8}, {2, 7}, {3, 6}, and {4, 5}. If five integers are selected from the first eight positive integers, by the pigeonhole principle at least two of them come from the same subset. Two such integers have a sum of 9, as desired. **b)** No. Take {1, 2, 3, 4}, for example. **15.** 4 **17.** 21,251 **19. a)** If there were fewer than 9 freshmen, fewer than 9 sophomores, and fewer than 9 juniors in the class, there would be no more than 8 with each of these three class standings, for a total of at most 24 students, contradicting the fact that there are 25 students in the class. **b)** If there were fewer than 3 freshmen, fewer than 19 sophomores, and fewer than 5 juniors, then there would be at most 2 freshmen, at most 18 sophomores, and at most 4 juniors, for a total of at most 24 students. This contradicts the fact that there are 25 students in the class. **21.** 4, 3, 2, 1, 8, 7, 6, 5, 12, 11, 10, 9, 16, 15, 14, 13 **23.** Number the seats around the table from 1 to 50, and think of seat 50 as being adjacent to seat 1. There are 25 seats with odd numbers and 25 seats with even numbers. If no more than 12 boys occupied the odd-numbered seats, then at least 13 boys would occupy the even-numbered seats, and vice versa. Without loss of generality, assume that at least 13 boys occupy the 25 odd-numbered seats. Then at least two of those boys must be in consecutive odd-numbered seats, and the person sitting between them will have boys as both of his or her neighbors.

25. **procedure** *long*($a_1, \ldots, a_n$: positive integers)
{first find longest increasing subsequence}
max := 0; *set* := 00 . . . 00 {n bits}
for i := 1 **to** 2^n
  last := 0; *count* := 0, *OK* := **true**
  for j := 1 **to** n
    if $set(j) = 1$ **then**
      if $a_j > last$ **then** *last* := a_j
      count := *count* + 1
    else *OK* := **false**
  if *count* > *max* **then**
    max := *count*
    best := *set*
  set := *set* + 1 (binary addition)
{*max* is length and *best* indicates the sequence}
{repeat for decreasing subsequence with only
  changes being $a_j < last$ instead of $a_j > last$
  and *last* := ∞ instead of *last* := 0}

27. By symmetry we need prove only the first statement. Let A be one of the people. Either A has at least four friends, or A has at least six enemies among the other nine people (because $3 + 5 < 9$). Suppose, in the first case, that B, C, D, and E are all A's friends. If any two of these are friends with each other, then we have found three mutual friends. Otherwise $\{B, C, D, E\}$ is a set of four mutual enemies. In the second case, let $\{B, C, D, E, F, G\}$ be a set of enemies of A. By Example 11, among B, C, D, E, F, and G there are either three mutual friends or three mutual enemies, who form, with A, a set of four mutual enemies. **29.** We need to show two things: that if we have a group of n people, then among them we must find either a pair of friends or a subset of n of them all of whom are mutual enemies; and that there exists a group of $n - 1$ people for which this is not possible. For the first statement, if there is any pair of friends, then the condition is satisfied, and if not, then every pair of people are enemies, so the second condition is satisfied. For the second statement, if we have a group of $n - 1$ people all of whom are enemies of each other, then there is neither a pair of friends nor a subset of n of them all of whom are mutual enemies. **31.** There are 6,432,816 possibilities for the three initials and a birthday. So, by the generalized pigeonhole principle, there are at least $\lceil 37{,}000{,}000/6{,}432{,}816 \rceil = 6$ people who share the same initials and birthday. **33.** Because $800{,}001 > 200{,}000$, the pigeonhole principle guarantees that there are at least two Parisians with the same number of hairs on their heads. The generalized pigeonhole principle guarantees that there are at least $\lceil 800{,}001/200{,}000 \rceil = 5$ Parisians with the same number of hairs on their heads. **35.** 18 **37.** Because there are six computers, the number of other computers a computer is connected to is an integer between 0 and 5, inclusive. However, 0 and 5 cannot both occur. To see this, note that if some computer is connected to no others, then no computer is connected to all five others, and if some computer is connected to all five others, then no computer is connected to no others. Hence, by the pigeonhole principle, because there are at most five possibilities for the number of computers a computer is connected to, there are at least two computers in the set of six connected to the same number of others. **39.** Label the computers C_1 through C_{100}, and label the printers P_1 through P_{20}. If we connect C_k to P_k for $k = 1, 2, \ldots, 20$ and connect each of the computers C_{21} through C_{100} to *all* the printers, then we have used a total of $20 + 80 \cdot 20 = 1620$ cables. Clearly this is sufficient, because if computers C_1 through C_{20} need printers, then they can use the printers with the same subscripts, and if any computers with higher subscripts need a printer instead of one or more of these, then they can use the printers that are not being used, because they are connected to all the printers. Now we must show that 1619 cables is not enough. Because there are 1619 cables and 20 printers, the average number of computers per printer is $1619/20$, which is less than 81. Therefore some printer must be connected to fewer than 81 computers. That means it is connected to 80 or fewer computers, so there are 20 computers that are not connected to it. If those 20 computers all needed a printer simultaneously, then they would be out of luck, because they are connected to at most the 19 other printers. **41.** Let a_i be the number of matches completed by hour i. Then $1 \le a_1 < a_2 < \cdots < a_{75} \le 125$. Also $25 \le a_1 + 24 < a_2 + 24 < \cdots < a_{75} + 24 \le 149$. There are 150 numbers $a_1, \ldots, a_{75}, a_1 + 24, \ldots, a_{75} + 24$. By the pigeonhole principle, at least two are equal. Because all the a_is are distinct and all the $(a_i + 24)$s are distinct, it follows that $a_i = a_j + 24$ for some $i > j$. Thus, in the period from the $(j+1)$st to the ith hour, there are exactly 24 matches. **43.** Use the generalized pigeonhole principle, placing the $|S|$ objects $f(s)$ for $s \in S$ in $|T|$ boxes, one for each element of T. **45.** Let d_j be $jx - N(jx)$, where $N(jx)$ is the integer closest to jx for $1 \le j \le n$. Each d_j is an irrational number between $-1/2$ and $1/2$. We will assume that n is even; the case where n is odd is messier. Consider the n intervals $\{x \mid j/n < x < (j+1)/n\}$, $\{x \mid -(j+1)/n < x < -j/n\}$ for $j = 0, 1, \ldots, (n/2) - 1$. If d_j belongs to the interval $\{x \mid 0 < x < 1/n\}$ or to the interval $\{x \mid -1/n < x < 0\}$ for some j, we are done. If not, because there are $n - 2$ intervals and n numbers d_j, the pigeonhole principle tells us that there is an interval $\{x \mid (k-1)/n < x < k/n\}$ containing d_r and d_s with $r < s$. The proof can be finished by showing that $(s - r)x$ is within $1/n$ of its nearest integer. **47. a)** Assume that $i_k \le n$ for all k. Then by the generalized pigeonhole principle, at least $\lceil (n^2+1)/n \rceil = n + 1$ of the numbers $i_1, i_2, \ldots, i_{n^2+1}$ are equal. **b)** If $a_{k_j} < a_{k_{j+1}}$, then the subsequence consisting of a_{k_j} followed by the increasing subsequence of length $i_{k_{j+1}}$ starting at $a_{k_{j+1}}$ contradicts the fact that $i_{k_j} = i_{k_{j+1}}$. Hence, $a_{k_j} > a_{k_{j+1}}$. **c)** If there is no increasing subsequence of length greater than n, then parts (a) and (b) apply. Therefore, we have $a_{k_{n+1}} > a_{k_n} > \cdots > a_{k_2} > a_{k_1}$, a decreasing sequence of length $n + 1$.

Section 6.3

1. *abc*, *acb*, *bac*, *bca*, *cab*, *cba* **3.** 720 **5. a)** 120 **b)** 720 **c)** 8 **d)** 6720 **e)** 40,320 **f)** 3,628,800 **7.** 15,120 **9.** 1320 **11. a)** 210 **b)** 386 **c)** 848 **d)** 252 **13.** $2(n!)^2$ **15.** 65,780 **17.** $2^{100} - 5051$ **19. a)** 1024 **b)** 45 **c)** 176 **d)** 252 **21. a)** 120 **b)** 24 **c)** 120 **d)** 24 **e)** 6 **f)** 0 **23.** 609,638,400 **25. a)** 94,109,400 **b)** 941,094 **c)** 3,764,376 **d)** 90,345,024 **e)** 114,072 **f)** 2328 **g)** 24 **h)** 79,727,040 **i)** 3,764,376 **j)** 109,440 **27. a)** 12,650 **b)** 303,600 **29. a)** 37,927 **b)** 18,915 **31. a)** 122,523,030 **b)** 72,930,375 **c)** 223,149,655 **d)** 100,626,625 **33.** 54,600 **35.** 45 **37.** 912 **39.** 11,232,000 **41.** $n!/(r(n-r)!)$ **43.** 13 **45.** 873

Section 6.4

1. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ **3.** $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$ **5.** 101 **7.** $-2^{10}\binom{19}{9} = -94{,}595{,}072$ **9.** $-2^{101}3^{99}\binom{200}{99}$ **11.** $(-1)^{(200-k)/3}\binom{100}{(200-k)/3}$ if $k \equiv 2 \pmod 3$ and $-100 \le k \le 200$; 0 otherwise **13.** 1 9 36 84 126 126 84 36 9 1 **15.** The sum of *all* the positive numbers $\binom{n}{k}$, as k runs from 0 to n, is 2^n, so each one of them is no bigger than this sum. **17.** $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2} \le \frac{n \cdot n \cdot \cdots \cdot n}{2 \cdot 2 \cdot \cdots \cdot 2} = n^k/2^{k-1}$ **19.** $\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k+1)!} \cdot [k + (n-k+1)] = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$ **21. a)** We show that each side counts the number of ways to choose from a set with n elements a subset with k elements and a distinguished element of that set. For the left-hand side, first choose the k-set (this can be done in $\binom{n}{k}$ ways) and then choose one of the k elements in this subset to be the distinguished element (this can be done in k ways). For the right-hand side, first choose the distinguished element out of the entire n-set (this can be done in n ways), and then choose the remaining $k - 1$ elements of the subset from the remaining $n - 1$ elements of the set (this can be done in $\binom{n-1}{k-1}$ ways). **b)** $k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}$ **23.** $\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!} = \frac{(n+1)}{k}\frac{n!}{(k-1)![n-(k-1)]!} = (n+1)\binom{n}{k-1}/k$. This identity together with $\binom{n}{0} = 1$ gives a recursive definition. **25.** $\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+1}{n+1} = \frac{1}{2}\left[\binom{2n+1}{n+1} + \binom{2n+1}{n+1}\right] = \frac{1}{2}\left[\binom{2n+1}{n+1} + \binom{2n+1}{n}\right] = \frac{1}{2}\binom{2n+2}{n+1}$ **27. a)** $\binom{n+r+1}{r}$ counts the number of ways to choose a sequence of r 0s and $n+1$ 1s by choosing the positions of the 0s. Alternately, suppose that the $(j+1)$st term is the last term equal to 1, so that $n \le j \le n+r$. Once we have determined where the last 1 is, we decide where the 0s are to be placed in the j spaces before the last 1. There are n 1s and $j-n$ 0s in this range. By the sum rule it follows that there are $\sum_{j=n}^{n+r}\binom{j}{j-n} = \sum_{k=0}^{r}\binom{n+k}{k}$ ways to do this. **b)** Let $P(r)$ be the statement to be proved. The basis step is the equation $\binom{n}{0} = \binom{n+1}{0}$, which is just $1 = 1$. Assume that $P(r)$ is true. Then $\sum_{k=0}^{r+1}\binom{n+k}{k} = \sum_{k=0}^{r}\binom{n+k}{k} + \binom{n+r+1}{r+1} = \binom{n+r+1}{r} + \binom{n+r+1}{r+1} = \binom{n+r+2}{r+1}$, using the inductive hypothesis