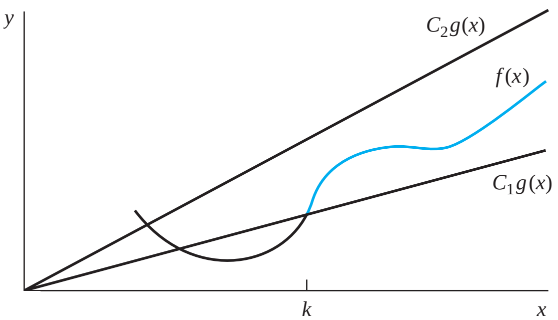
set woman j's proposal list to be empty
while rejected men remain
for $i := 1$ **to** s
if man i is marked rejected **then** add i to the proposal list for the woman j who ranks highest on his preference list but does not appear on his rejection list, and mark i as not rejected
for $j := 1$ **to** s
if woman j's proposal list is nonempty **then** remove from j's proposal list all men i except the man i_0 who ranks highest on her preference list, and for each such man i mark him as rejected and add j to his rejection list
for $j := 1$ **to** s
match j with the one man on j's proposal list
{This matching is stable.}

63. If the assignment is not stable, then there is a man m and a woman w such that m prefers w to the woman w' with whom he is matched, and w prefers m to the man with whom she is matched. But m must have proposed to w before he proposed to w', because he prefers the former. Because m did not end up matched with w, she must have rejected him. Women reject a suitor only when they get a better proposal, and they eventually get matched with a pending suitor, so the woman with whom w is matched must be better in her eyes than m, contradicting our original assumption. Therefore the marriage is stable. **65.** Run the two programs on their inputs concurrently and report which one halts.

Section 3.2

1. The choices of C and k are not unique. **a)** $C = 1, k = 10$ **b)** $C = 4, k = 7$ **c)** No **d)** $C = 5, k = 1$ **e)** $C = 1, k = 0$ **f)** $C = 1, k = 2$ **3.** $x^4 + 9x^3 + 4x + 7 \le 4x^4$ for all $x > 9$; witnesses $C = 4, k = 9$ **5.** $(x^2 + 1)/(x + 1) = x - 1 + 2/(x + 1) < x$ for all $x > 1$; witnesses $C = 1, k = 1$ **7.** The choices of C and k are not unique. **a)** $n = 3, C = 3, k = 1$ **b)** $n = 3, C = 4, k = 1$ **c)** $n = 1, C = 2, k = 1$ **d)** $n = 0, C = 2, k = 1$ **9.** $x^2 + 4x + 17 \le 3x^3$ for all $x > 17$, so $x^2 + 4x + 17$ is $O(x^3)$, with witnesses $C = 3, k = 17$. However, if x^3 were $O(x^2 + 4x + 17)$, then $x^3 \le C(x^2 + 4x + 17) \le 3Cx^2$ for some C, for all sufficiently large x, which implies that $x \le 3C$ for all sufficiently large x, which is impossible. Hence, x^3 is not $O(x^2 + 4x + 17)$. **11.** $3x^4 + 1 \le 4x^4 = 8(x^4/2)$ for all $x > 1$, so $3x^4 + 1$ is $O(x^4/2)$, with witnesses $C = 8, k = 1$. Also $x^4/2 \le 3x^4 + 1$ for all $x > 0$, so $x^4/2$ is $O(3x^4 + 1)$, with witnesses $C = 1, k = 0$. **13.** Because $2^n \le 3^n$ for all $n > 0$, it follows that 2^n is $O(3^n)$, with witnesses $C = 1, k = 0$. However, if 3^n were $O(2^n)$, then for some C, $3^n \le C \cdot 2^n$ for all sufficiently large n. This says that $C \ge (3/2)^n$ for all sufficiently large n, which is impossible. Hence, 3^n is not $O(2^n)$. **15.** All functions for which there exist real numbers k and C with $|f(x)| \le C$ for $x > k$. These are the functions $f(x)$ that are bounded for all sufficiently large x. **17.** There are constants C_1, C_2, k_1, and k_2 such that $|f(x)| \le C_1|g(x)|$ for all $x > k_1$ and $|g(x)| \le C_2|h(x)|$ for all $x > k_2$. Hence, for $x > \max(k_1, k_2)$ it follows that $|f(x)| \le C_1|g(x)| \le C_1C_2|h(x)|$. This shows that $f(x)$ is $O(h(x))$. **19.** 2^{n+1} is $O(2^n)$; 2^{2n} is not. **21.** $1000 \log n$, $\sqrt{n}$, $n \log n$, $n^2/1000000$, 2^n, 3^n, $2n!$ **23.** The algorithm that uses $n \log n$ operations **25. a)** $O(n^3)$ **b)** $O(n^5)$ **c)** $O(n^3 \cdot n!)$ **27. a)** $O(n^2 \log n)$ **b)** $O(n^2(\log n)^2)$ **c)** $O(n^{2^n})$ **29. a)** Neither $\Theta(x^2)$ nor $\Omega(x^2)$ **b)** $\Theta(x^2)$ and $\Omega(x^2)$ **c)** Neither $\Theta(x^2)$ nor $\Omega(x^2)$ **d)** $\Omega(x^2)$, but not $\Theta(x^2)$ **e)** $\Omega(x^2)$, but not $\Theta(x^2)$ **f)** $\Omega(x^2)$ and $\Theta(x^2)$ **31.** If $f(x)$ is $\Theta(g(x))$, then there exist constants C_1 and C_2 with $C_1|g(x)| \le |f(x)| \le C_2|g(x)|$. It follows that $|f(x)| \le C_2|g(x)|$ and $|g(x)| \le (1/C_1)|f(x)|$ for $x > k$. Thus, $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$. Conversely, suppose that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$. Then there are constants C_1, C_2, k_1, and k_2 such that $|f(x)| \le C_1|g(x)|$ for $x > k_1$ and $|g(x)| \le C_2|f(x)|$ for $x > k_2$. We can assume that $C_2 > 0$ (we can always make C_2 larger). Then we have $(1/C_2)|g(x)| \le |f(x)| \le C_1|g(x)|$ for $x > \max(k_1, k_2)$. Hence, $f(x)$ is $\Theta(g(x))$. **33.** If $f(x)$ is $\Theta(g(x))$, then $f(x)$ is both $O(g(x))$ and $\Omega(g(x))$. Hence, there are positive constants C_1, k_1, C_2, and k_2 such that $|f(x)| \le C_2|g(x)|$ for all $x > k_2$ and $|f(x)| \ge C_1|g(x)|$ for all $x > k_1$. It follows that $C_1|g(x)| \le |f(x)| \le C_2|g(x)|$ whenever $x > k$, where $k = \max(k_1, k_2)$. Conversely, if there are positive constants C_1, C_2, and k such that $C_1|g(x)| \le |f(x)| \le C_2|g(x)|$ for $x > k$, then taking $k_1 = k_2 = k$ shows that $f(x)$ is both $O(g(x))$ and $\Theta(g(x))$.

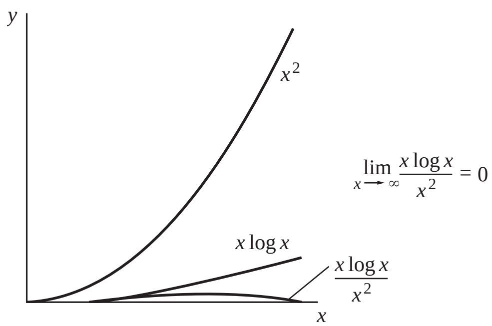
35.



37. If $f(x)$ is $\Theta(1)$, then $|f(x)|$ is bounded between positive constants C_1 and C_2. In other words, $f(x)$ cannot grow larger than a fixed bound or smaller than the negative of this bound and must not get closer to 0 than some fixed bound. **39.** Because $f(x)$ is $O(g(x))$, there are constants C and k such that $|f(x)| \le C|g(x)|$ for $x > k$. Hence, $|f^n(x)| \le C^n|g^n(x)|$ for $x > k$, so $f^n(x)$ is $O(g^n(x))$ by taking the constant to be C^n. **41.** Because $f(x)$ and $g(x)$ are increasing and unbounded, we can assume $f(x) \ge 1$ and $g(x) \ge 1$ for sufficiently large x. There are constants C and k with $f(x) \le Cg(x)$ for $x > k$. This implies that $\log f(x) \le \log C + \log g(x) < 2 \log g(x)$ for sufficiently large x. Hence, $\log f(x)$ is $O(\log g(x))$. **43.** By definition there are positive constraints C_1, C_1', C_2, C_2', k_1, k_1', k_2, and k_2' such that $f_1(x) \ge C_1|g(x)|$ for all $x > k_1$, $f_1(x) \le C_1'|g(x)|$ for all $x > k_1'$, $f_2(x) \ge C_2|g(x)|$ for all $x > k_2$, and $f_2(x) \le C_2'|g(x)|$ for all $x > k_2'$. Adding the first and third inequalities shows that $f_1(x) + f_2(x) \ge (C_1 + C_2)|g(x)|$ for all $x > k$ where

$k = \max(k_1, k_2)$. Adding the second and fourth inequalities shows that $f_1(x) + f_2(x) \le (C_1' + C_2')|g(x)|$ for all $x > k'$ where $k' = \max(k_1', k_2')$. Hence, $f_1(x) + f_2(x)$ is $\Theta(g(x))$. This is no longer true if f_1 and f_2 can assume negative values. **45.** This is false. Let $f_1 = x^2 + 2x$, $f_2(x) = x^2 + x$, and $g(x) = x^2$. Then $f_1(x)$ and $f_2(x)$ are both $O(g(x))$, but $(f_1 - f_2)(x)$ is not. **47.** Take $f(n)$ to be the function with $f(n) = n$ if n is an odd positive integer and $f(n) = 1$ if n is an even positive integer and $g(n)$ to be the function with $g(n) = 1$ if n is an odd positive integer and $g(n) = n$ if n is an even positive integer. **49.** There are positive constants $C_1, C_2, C_1', C_2', k_1, k_1', k_2$, and k_2' such that $|f_1(x)| \ge C_1|g_1(x)|$ for all $x > k_1$, $|f_1(x)| \le C_1'|g_1(x)|$ for all $x \ge k_1'$, $|f_2(x)| > C_2|g_2(x)|$ for all $x > k_2$, and $|f_2(x)| \le C_2'|g_2(x)|$ for all $x > k_2'$. Because f_2 and g_2 are never zero, the last two inequalities can be rewritten as $|1/f_2(x)| \le (1/C_2)|1/g_2(x)|$ for all $x > k_2$ and $|1/f_2(x)| \ge (1/C_2')|1/g_2(x)|$ for all $x > k_2'$. Multiplying the first and rewritten fourth inequalities shows that $|f_1(x)/f_2(x)| \ge (C_1/C_2')|g_1(x)/g_2(x)|$ for all $x > \max(k_1, k_2')$, and multiplying the second and rewritten third inequalities gives $|f_1(x)/f_2(x)| \le (C_1'/C_2)|g_1(x)/g_2(x)|$ for all $x > \max(k_1', k_2)$. It follows that f_1/f_2 is big-Theta of g_1/g_2. **51.** There exist positive constants C_1, C_2, k_1, k_2, k_1', k_2' such that $|f(x, y)| \le C_1|g(x, y)|$ for all $x > k_1$ and $y > k_2$ and $|f(x, y)| \ge C_2|g(x, y)|$ for all $x > k_1'$ and $y > k_2'$. **53.** $(x^2 + xy + x \log y)^3 < (3x^2y^3) = 27x^6y^3$ for $x > 1$ and $y > 1$, because $x^2 < x^2y$, $xy < x^2y$, and $x \log y < x^2y$. Hence, $(x^2 + xy + x \log y)^3$ is $O(x^6y^3)$. **55.** For all positive real numbers x and y, $\lfloor xy \rfloor \le xy$. Hence, $\lfloor xy \rfloor$ is $O(xy)$ from the definition, taking $C = 1$ and $k_1 = k_2 = 0$. **57.** Clearly $n^d < n^c$ for all $n \ge 2$; therefore n^d is $O(n^c)$. The ratio $n^d/n^c = n^{d-c}$ is unbounded so there is no constant C such that $n^d \le Cn^c$ for large n. **59.** If f and g are positive-valued functions such that $\lim_{n\to\infty} f(x)/g(x) = C < \infty$, then $f(x) < (C+1)g(x)$ for large enough x, so $f(n)$ is $O(g(n))$. If that limit is ∞, then clearly $f(n)$ is not $O(g(n))$. Here repeated applications of L'Hôpital's rule shows that $\lim_{x\to\infty} x^d/b^x = 0$ and $\lim_{x\to\infty} b^x/x^d = \infty$. **61. a)** $\lim_{x\to\infty} x^2/x^3 = \lim_{x\to\infty} 1/x = 0$ **b)** $\lim_{x\to\infty} \frac{x \log x}{x^2} = \lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1}{x \ln 2} = 0$ (using L'Hôpital's rule) **c)** $\lim_{x\to\infty} \frac{x^2}{2^x} = \lim_{x\to\infty} \frac{2x}{2^x \cdot \ln 2} = \lim_{x\to\infty} \frac{2}{2^x \cdot (\ln 2)^2} = 0$ (using L'Hôpital's rule) **d)** $\lim_{x\to\infty} \frac{x^2+x+1}{x^2} = \lim_{x\to\infty} \left(1 + \frac{1}{x} + \frac{1}{x^2}\right) = 1 \ne 0$

63.



65. No. Take $f(x) = 1/x^2$ and $g(x) = 1/x$. **67. a)** Because $\lim_{x\to\infty} f(x)/g(x) = 0$, $|f(x)|/|g(x)| < 1$ for sufficiently large x. Hence, $|f(x)| < |g(x)|$ for $x > k$ for some constant k. Therefore, $f(x)$ is $O(g(x))$. **b)** Let $f(x) = g(x) = x$. Then $f(x)$ is $O(g(x))$, but $f(x)$ is not $o(g(x))$ because $f(x)/g(x) = 1$. **69.** Because $f_2(x)$ is $o(g(x))$, from Exercise 67(a) it follows that $f_2(x)$ is $O(g(x))$. By Corollary 1, we have $f_1(x) + f_2(x)$ is $O(g(x))$. **71.** We can easily show that $(n-i)(i+1) \ge n$ for $i = 0, 1, \ldots, n-1$. Hence, $(n!)^2 = (n \cdot 1)((n-1) \cdot 2) \cdot ((n-2) \cdot 3) \cdots (2 \cdot (n-1)) \cdot (1 \cdot n) \ge n^n$. Therefore, $2 \log n! \ge n \log n$. **73.** Compute that $\log 5! \approx 6.9$ and $(5 \log 5)/4 \approx 2.9$, so the inequality holds for $n = 5$. Assume $n \ge 6$. Because $n!$ is the product of all the integers from n down to 1, we have $n! > n(n-1)(n-2) \cdots \lceil n/2 \rceil$ (because at least the term 2 is missing). Note that there are more than $n/2$ terms in this product, and each term is at least as big as $n/2$. Therefore the product is greater than $(n/2)^{(n/2)}$. Taking the log of both sides of the inequality, we have $\log n! > \log\left(\frac{n}{2}\right)^{n/2} = \frac{n}{2} \log \frac{n}{2} = \frac{n}{2}(\log n - 1) > (n \log n)/4$, because $n > 4$ implies $\log n - 1 > (\log n)/2$. **75.** All are not asymptotic.

Section 3.3

1. $O(1)$ **3.** $O(n^2)$ **5.** $2n - 1$ **7.** Linear **9.** $O(n)$
11. a) procedure *disjointpair*$(S_1, S_2, \ldots, S_n$: subsets of $\{1, 2, \ldots, n\})$

```
answer := false
for i := 1 to n
   for j := i + 1 to n
      disjoint := true
      for k := 1 to n
         if k ∈ S_i and k ∈ S_j then disjoint := false
      if disjoint then answer := true
return answer
```

b) $O(n^3)$ **13. a)** *power* := 1, y := 1; i := 1, *power* := 2, y := 3; i := 2, *power* := 4, y := 15 **b)** $2n$ multiplications and n additions **15. a)** $2^{10^9} \approx 10^{3 \times 10^8}$ **b)** 10^9 **c)** 3.96×10^7 **d)** 3.16×10^4 **e)** 29 **f)** 12 **17. a)** $2^{2^{60 \cdot 10^{12}}}$ **b)** $2^{60 \cdot 10^{12}}$ **c)** $\lfloor 2^{\sqrt{60 \cdot 10^6}} \rfloor \approx 2 \times 10^{2331768}$ **d)** 60,000,000 **e)** 7,745,966 **f)** 45 **g)** 6 **19. a)** 36 years **b)** 13 days **c)** 19 minutes **21. a)** Less than 1 millisecond more **b)** 100 milliseconds more **c)** $2n + 1$ milliseconds more **d)** $3n^2 + 3n + 1$ milliseconds more **e)** Twice as much time **f)** 2^{2n+1} times as many milliseconds **g)** $n + 1$ times as many milliseconds **23.** The average number of comparisons is $(3n + 4)/2$. **25.** $O(\log n)$ **27.** $O(n)$ **29.** $O(n^2)$ **31.** $O(n)$ **33.** $O(n)$ **35.** $O(\log n)$ comparisons; $O(n^2)$ swaps **37.** $O(n^2 2^n)$ **39. a)** doubles **b)** increases by 1 **41.** Use Algorithm 1, where **A** and **B** are now $n \times n$ upper triangular matrices, by replacing m by n in line 1, and