

Discrete Mathematics

ECS 20 (Fall 2016)

Patrice Koehl
koehl@cs.ucdavis.edu

September 28, 2016

Homework 1 - For 9/28/2016

Exercise 1

A ball and a bat cost \$11.10 (total). The bat costs \$10.0 more than the ball. How much does the ball cost?

Intuitively, we would say the bat costs \$11.0 and the ball costs \$1.10 but we would be wrong! (admit it that this was your first reaction to this problem!). We need a more formal approach to solve this problem. Let x be the cost of the ball, and y the cost of the bat. We translate the two sentences given to us as:

$$\begin{aligned}x + y &= 11.10 \\ y &= x + 10.0\end{aligned}$$

The solution to this system of equations is $x = \$0.55$ and $y = \$10.55$.

Exercise 2

Prove the following statements:

- a) The sum of any three consecutive odd numbers is always a multiple of 3

Let N be an odd number. There exists an integer number k such that $n = 2k + 1$. The two odd numbers that follows N are $N + 2$ and $N + 4$, which can be rewritten as $2k + 3$ and $2k + 5$. Let S be the sum of these three consecutive odd numbers. Then:

$$\begin{aligned}S &= N + N + 2 + N + 4 \\ &= 2k + 1 + 2k + 3 + 2k + 5 \\ &= 6k + 9 \\ &= 3(2k + 3)\end{aligned}$$

As $2k + 3$ is an integer, S is a multiple of 3. As this is true for all values of N , the proposition is always true.

- a) The sum of any five consecutive even numbers is always a multiple of 10

Let N be an even number. There exists an integer number k such that $n = 2k$. The four even numbers that follows N are $N + 2$, $N + 4$, $N + 6$ and $N + 8$ which can be rewritten as $2k + 2$, $2k + 4$, $2k + 6$, and $2k + 8$. Let S be the sum of these five consecutive even numbers. Then:

$$\begin{aligned} S &= N + N + 2 + N + 4 + N + 6 + N + 8 \\ &= 2k + 2k + 2 + 2k + 4 + 2k + 6 + 2k + 8 \\ &= 10k + 20 \\ &= 10(k + 2) \end{aligned}$$

As $k + 2$ is an integer, S is a multiple of 10. As this is true for all values of N , the proposition is always true.

- c) Prove that if you add the squares of three consecutive integer numbers and then subtract two, you always get a multiple of 3.

Let N be an integer number. The two numbers that follows N are $N + 1$ and $N + 2$. Let S be the sum of the squares of these three consecutive numbers. Then:

$$\begin{aligned} S &= N^2 + (N + 1)^2 + (N + 2)^2 \\ &= N^2 + N^2 + 2N + 1 + N^2 + 4N + 4 \\ &= 3N^2 + 6N + 5 \end{aligned}$$

Therefore,

$$S - 2 = 3(N^2 + 2N + 1)$$

As $3(N^2 + 2N + 1)$ is an integer, $S - 2$ is a multiple of 3. As this is true for all values of N , the proposition is always true.

Exercise 3

Roger is an amateur magician. In one of his tricks he invites people in the audience to think of a number (integer). He then asks them to carry out the following simple instructions:

triple your number
then add 5
then multiply the number you now have by itself
then subtract 25
then divide by 3
then divide by your original number

Let N be the initial integer number. Let us apply the set of instructions and see what comes out:

Instruction	Input	Result
double the number	N	$3N$
add 5	$3N$	$3N + 5$
multiply by itself	$3N + 5$	$9N^2 + 30N + 25$
subtract 25	$9N^2 + 30N + 25$	$9N^2 + 30N$
divide by 3	$9N^2 + 30N$	$3N^2 + 10N$
divide by the original number	$3N^2 + 10N$	$3N + 10$

The final number is simply $3N + 10$. By subtracting 10 and dividing by 3, we get back the original number in all cases. No magic there.

Exercise 4

Prove the following identities for p, q, m, n , and x real numbers:

a) $8(p - q) + 3(p + q) = 2(p + 2q) + 9(p - q)$

Let p and q be two real numbers, and let $LHS = 8(p - q) + 3(p + q)$ and $RHS = 2(p + 2q) + 9(p - q)$. Then:

$$\begin{aligned} LHS &= 8p - 8q + 3p + 3q \\ &= 11p - 5q \end{aligned}$$

and

$$\begin{aligned} RHS &= 2p + 4q + 9p - 9q \\ &= 11p - 5q \end{aligned}$$

Therefore $LHS = RHS$ for all p and q , and the identity is true.

b) $x(m + n) + y(n - m) = m(x - y) + n(x + y)$

Let x, y, m and n be four real numbers, and let $LHS = x(m + n) + y(n - m)$ and $RHS = m(x - y) + n(x + y)$. Then:

$$LHS = xm + xn + yn - ym$$

and

$$RHS = xm + xn + yn - ym$$

Therefore $LHS = RHS$ for all x, y, n and m , and the identity is true.

c) $(x + 2)(x + 10) - (x - 5)(x - 4) = 21x$

Let x be a real number and let $LHS = (x + 2)(x + 10) - (x - 5)(x - 4)$ and $RHS = 21x$. Then:

$$\begin{aligned}
LHS &= x^2 + 10x + 2x + 20 - x^2 + 4x + 5x - 20 \\
&= 21x \\
&= RHS
\end{aligned}$$

The identity is true for all x .

d) $m^4 - 1 = (m^2 - 1)(m^2 + 1)$

Let m be a real number and let $LHS = m^4 - 1$ and $RHS = (m^2 - 1)(m^2 + 1)$. Then

$$\begin{aligned}
LHS &= (m^2)^2 - 1^2 \\
&= (m^2 - 1)(m^2 + 1) \\
&= RHS
\end{aligned}$$

The identity is true for all m .

Extra credit

Four persons need to cross a bridge to get back to their camp at night. Unfortunately, they only have one flashlight and it only has enough light left for seventeen minutes. The bridge is too dangerous to cross without a flashlight, and it is strong enough to support only two persons at any given time. The flashlight cannot be thrown from one side of the bridge to the other. Each of the campers walks at a different speed. One can cross the bridge in 1 minute, another in 2 minutes, the third in 5 minutes, and the last one takes 10 minutes to cross. Tell these people how they can make it across in 17 minutes.

There are several ways to solve this problems. let A , B , C , and D be the persons that can cross the bridge in 1, 2, 5, and 10 minutes, respectively. Let S be on one side of the bridge, and T on the other side. A solution to the problem needs to alternate trips from $S \rightarrow T$ with trips from $T \rightarrow S$. We build the following table:

Trip	Participants	Trip duration	Total time
$S \rightarrow T$	A and B	2	2
$T \rightarrow S$	A	1	3
$S \rightarrow T$	C and D	10	13
$T \rightarrow S$	B	2	15
$S \rightarrow T$	A and B	2	17