Homework 3: Solutions

ECS 20 (Fall 2016)

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Exercise 1

Show that this implication is a tautology, by using a table of truth: $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$.

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	A:$[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)]$	$A \rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Exercise 2

Show that $[(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology

p	q	r	$p \vee q$	$\neg p \vee r$	A:$(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	$A \rightarrow (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	F	T	T
F	F	F	F	T	F	F	T

Exercise 3

a) Let x be a real number. Show that " if x^2 is irrational, it follows that x is irrational."

Let $p : x^2$ is irrational, and let $q : x$ is irrational. We need to prove that $p \rightarrow q$. We use an indirect proof, i.e. we show that $\neg q \rightarrow \neg p$.

Let us assume $\neg q$, i.e. x is rational. There exists an integer a and a non-zero integer b such that $x = \frac{a}{b}$. Then $x^2 = \frac{a^2}{b^2}$. Since a^2 and b^2 are integers, x^2 is a rational number. Therefore $\neg p$ is true. Therefore $\neg q \rightarrow \neg p$ is true, and consequently $p \rightarrow q$ is true.

b) Based on question a), can you say that " if x is irrational, it follows that x^2 is irrational."

It is not a valid argument. The statement in a) can be simplified as "$p \rightarrow q$, while the second statement is the converse of the first statement: they are not equivalent.

Exercise 4

Prove that a square of an integer ends with a 0, 1, 4, 5 6 or 9. (Hint: let $n = 10k + l$, where $l = 0$, 1, ,9)

Let n bean integer; there exists two integers k and l such that $n = 10k + l$ where $0 \leq l \leq 9$. We get:

$$\begin{aligned} n^2 &= (10k + l)^2 \\ &= 100k + 20kl + l^2 \\ &= k \times 100 + 2kl \times 10 + l^2 \end{aligned}$$

$k \times 100$ and $2kl \times 10$ are multiples of 10. Therefore, n^2 ends as l^2. In the following table, we show that l^2 always end with a 0, 1, 4, 5, 6, or 9.

l	l^2	end
0	0	0
1	1	1
2	4	4
3	9	9
4	16	6
5	25	5
6	36	6
7	49	9
8	64	4
9	81	1

Exercise 5

Prove that if n is a positive integer, then n is even if and only if $5n + 6$ is even.

Let p be the proposition "n is even" and q be the proposition "$5n + 6$ is even". We want to show that $p \leftrightarrow q$, which is logically equivalent to show that $p \rightarrow q$ and $q \rightarrow p$.

i) Let us show $p \rightarrow q$:

Hypothesis: p is true, i.e. n is even. As n is even, there exists an integer k such that $n = 2k$. We get:

$$\begin{aligned} 5n+6 &= 5(2k)+6 \\ &= 10k+6 \\ &= 2\times(5k+3) \end{aligned}$$

Since $5k+3$ is an integer, $5n+6$ is a multiple of 2: it is even.

ii) Let us show $q \to p$:

Hypothesis: q is true, i.e. $5n+6$ is even. As $5n+6$ is even, there exists an integer k such that $5n+6=2k$. We get:

$$\begin{aligned} 5n &= 2k-6 \\ n &= 2k-6-4n \\ n &= 2\times(k-3-2n) \end{aligned}$$

Since $k-3-2n$ is an integer, n is a multiple of 2: it is even.

We conclude: n is even $\leftrightarrow$ $5n+6$ is even.

Exercise 6

Prove that either $3\times 100^{450}+15$ or $3\times 100^{450}+16$ is not a perfect square.

Let $n=3\times 100^{450}+15$. The two numbers are n and $n+1$.

Proof by contradiction: Let us suppose that both n and $n+1$ are perfect squares:

$$\exists k\in\mathbb{Z}, k^2=n$$
$$\exists l\in\mathbb{Z}, l^2=n+1$$

Then

$$\begin{aligned} l^2 &= k^2+1 \\ (l-k)(l+k) &= 1 \end{aligned}$$

Since l and k are integers, there are only two cases:

- $l-k=1$ and $l+k=1$, i.e. $l=1$ and $k=0$. Then we would have $k^2=0$, i.e. $n=0$: contradiction
- $l-k=-1$ and $l+k=-1$, i.e. $l=-1$ and $k=0$. Again, contradiction.

We can conclude that the proposition is true.

Exercise 7

Prove or disprove that if a and b are rational numbers, then a^b is also rational.

It is not true. Let $a=2$ and $b=1/2$, both a and b are rational numbers. However, $a^b=2^{\frac{1}{2}}=\sqrt{2}$ which is not rational (see lecture notes).

Exercise 8

Prove that at least one of the real numbers $a_1, a_2, \ldots a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

We use a proof by contradiction.

Suppose none of the real numbers a_1, a_2, ..., a_n is greater than or equal to the average of these numbers, denoted by $\overline{a}$.

By definition

$$\overline{a} = \frac{a_1 + a_2 + ... + a_n}{n}$$

Our hypothesis is that:

$$\begin{aligned} a_1 &< \overline{a} \\ a_2 &< \overline{a} \\ ... &< ... \\ a_n &< \overline{a} \end{aligned}$$

We sum up all these equations and get the following:

$$a_1 + a_2 + ... + a_n < n * \overline{a}$$

Replacing $\overline{a}$ in equation (9) by its value given in equation (4) we get:

$$a_1 + a_2 + ... + a_n < a_1 + a_2 + ... + a_n$$

This is not possible: a number cannot be strictly smaller than itself: we have reached a contradition. Therefore our hypothesis was wrong, and the original statement was correct.

Exercise 9

The correct order is: 3, 5, 4, 2, 1.

Exercise 10

Prove that these four statements are equivalent: (i) n^2 is odd, (ii) $1-n$ is even, (iii) n^3 is odd, (iv) n^2+1 is even.

Let us define the four propositions:

- p : n^2 is odd
- q : $1-n$ is even
- r : n^3 is odd
- s : n^2+1 is even

we will show:

- $q \leftrightarrow p$
- $q \leftrightarrow r$
- $q \leftrightarrow s$

If these three logical equivalence are true, all four propositions are equivalent.

1) **Proof 1**: $1-n$ is even $\leftrightarrow n^2$ is odd.

   We need to show two implications: (1) $1-n$ is even implies n^2 is odd and (2), n^2 is odd implies that $1-n$ is even.

   a) **Implication 1**: $q \rightarrow p$
   We use a direct proof.
   Hypothesis: q is true, i.e. $1-n$ is even. There exists an integer k such that $1-n=2k$. Therefore $n=1-2k$. Taking the squares on each side, we get:

   $$n^2 = (1-2k)^2 = 4k^2 - 2k + 1 = 2(2k^2 - k) + 1$$

   Therefore n^2 is odd. We conclude that $q \rightarrow p$.

   b) **Implication 2**: $p \rightarrow q$.
   We use an indirect proof, i.e. we show that: $\neg q \rightarrow \neg p$.
   * $\neg q$: $1-n$ is odd
   * $\neg p$: n^2 is even.

   Let us suppose that $1-n$ is odd. There exists an integer k such that $1-n=2k+1$; therefore $n=-2k$. Taking the square, we find that $n^2=4k^2$, and therefore n^2 is even. We conclude that $\neg q \rightarrow \neg p$; its contrapositive is then also true, i.e. $p \rightarrow q$.

   We conclude that $q \rightarrow p$ and $p \rightarrow q$, and therefore $p \Leftrightarrow q$.

2) **Proof 2**: $1-n$ is even $\leftrightarrow n^3$ is odd.

   We need to show two implications: (1) $1-n$ is even implies n^2 is odd and (2), n^2 is odd implies that $1-n$ is even.

   a) **Implication 1**: $q \rightarrow r$
   We use a direct proof.
   Hypothesis: q is true, i.e. $1-n$ is even. There exists an integer k such that $1-n=2k$. Therefore $n=1-2k$. Taking the cubes on each side, we get:

   $$n^3 = (1-2k)^3 = -8k^3 + 12k^2 - 6k + 1 = 2(-4k^3 + 6k^2 - 3k) + 1$$

   Therefore n^3 is odd. We conclude that $q \rightarrow r$.

   b) **Implication 2**: $r \rightarrow q$.
   We use an indirect proof, i.e. we show that: $\neg q \rightarrow \neg r$.
   * $\neg q$: $1-n$ is odd
   * $\neg p$: n^3 is even.

Let us suppose that $1-n$ is odd. There exists an integer k such that $1-n=2k+1$; therefore $n=-2k$. Taking the cube, we find that $n^3=8k^3=2(4k^3)$, and therefore n^3 is even.

We conclude that $\neg q \to \neg r$; its contrapositive is then also true, i.e. $r \to q$.

We conclude that $q \to r$ and $r \to q$, and therefore $r \Leftrightarrow q$.

1) **Proof 3**: $1-n$ is even $\leftrightarrow n^2+1$ is even.

We need to show two implications: (1) $1-n$ is even implies n^2+1 is even and (2), n^2+1 is even implies that $1-n$ is even.

This is nearly a copy of proof 1!!

a) **Implication 1**: $q \to s$

We use a direct proof.

Hypothesis: q is true, i.e. $1-n$ is even. There exists an integer k such that $1-n=2k$. Therefore $n=1-2k$. Taking the squares on each side, we get:

$$n^2=(1-2k)^2=4k^2-2k+1=2(2k^2-k)+1$$

Therefore:

$$n^2+1=2(2k^2-k)+1+1=2*(2k^2-k+1)$$

Therefore n^2+1 is even. We conclude that $q \to s$.

b) **Implication 2**: $s \to q$.

We use an indirect proof, i.e. we show that: $\neg q \to \neg s$.

* $\neg q$: $1-n$ is odd
* $\neg s$: n^2+1 is odd.

Let us suppose that $1-n$ is odd. There exists an integer k such that $1-n=2k+1$; therefore $n=-2k$. Taking the square, we find that $n^2=4k^2$, and therefore $n^2+1=4k^2+1$, i.e. n^2+1 is odd.

We conclude that $\neg q \to \neg s$; its contrapositive is then also true, i.e. $s \to q$.

We conclude that $q \to s$ and $p \to s$, and therefore $s \Leftrightarrow q$.

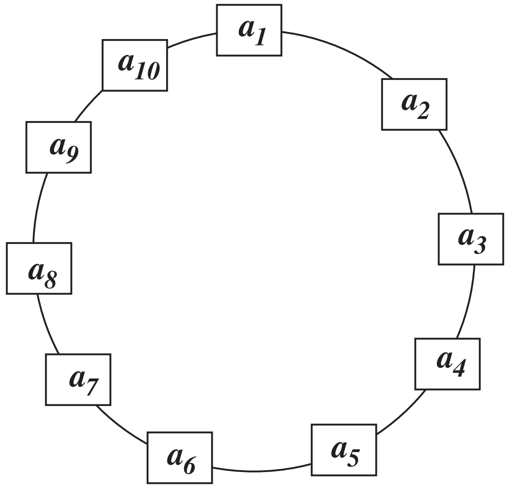
Extra Credit

Use Exercise 8 to show that if the first 10 strictly positive integers are placed around a circle, in any order, then there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

Let a_1, a_2, ..., a_{10} be an arbitrary order of 10 positive integers from 1 to 10 being placed around a circle:

Since the ten numbers a correspond to the first 10 positive integers, we get:

$$a_1+a_2+...+a_{10} \quad = \quad 1+2+...+10=55 \tag{1}$$



Notice that the a_1, a_2, ..., a_{10} are not necessarily in the order 1, 2, ..., 10. They do include however the ten integers from 1 to 10: these is why the sum is 55

Let us now consider the different sums S_i of three consecutive sites around the circle. There are 10 such sums:

$$\begin{aligned}
S_1 &= a_1 + a_2 + a_3 \\
S_2 &= a_2 + a_3 + a_4 \\
S_3 &= a_3 + a_4 + a_5 \\
S_4 &= a_4 + a_5 + a_6 \\
S_5 &= a_5 + a_6 + a_7 \\
S_6 &= a_6 + a_7 + a_8 \\
S_7 &= a_7 + a_8 + a_9 \\
S_8 &= a_8 + a_9 + a_{10} \\
S_9 &= a_9 + a_{10} + a_1 \\
S_{10} &= a_{10} + a_1 + a_2
\end{aligned}$$

We do not know the values of the individual sums S_i; however, we can compute the sum of these numbers:

$$\begin{aligned}
S_1 + S_2 + ... + S_{10} &= (a_1 + a_2 + a_3) + (a_2 + a_3 + a_4) + ... + (a_{10} + a_1 + a_2) \\
&= 3 * (a_1 + a_2 + ... + a_{10}) \\
&= 3 * 55 \\
&= 165
\end{aligned}$$

The average of S_1, S_2, ..., S_{10} is therefore:

$$\begin{aligned}\overline{S} &= \frac{S_1 + S_2 + ... + S_{10}}{10} \\ &= \frac{165}{10} \\ &= 16.5\end{aligned}$$

Based on the conclusion of Exercise 8, at least one of S_1, S_2, ..., S_{10} is greater to or equal to $\overline{S}$, i.e., 16.5. Because S_1, S_2, ..., S_{10} are all integers, they cannot be equal to 16.5. Thus, at least one of S_1, S_2, ..., S_{10} is greater to or equal to 17.