Homework 4 Solutions

ECS 20 (Fall 2016)

Patrice Koehl
koehl@cs.ucdavis.edu

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Exercise 1

Let p be the proposition: "n is even", and q be the proposition "$n + 4$ is even".

- **Direct proof**: We show directly $p \rightarrow q$.
  Hypothesis: p is true, i.e. n is even.

  If n is even, there exists an integer k such that $n = 2 * k$, Then, $n + 4 = 2 * k + 4 = 2 * (k + 2)$. Therefore, $n + 4$ is even, i.e. q is true. We conclude that $p \rightarrow q$.

- **Indirect proof**: We show that $\neg q \rightarrow \neg p$.
  Hypothesis: $\neg q$ is true, i.e. $n{+}4$ is odd. Then there exists an integer k such that $n{+}4 = 2{*}k{+}1$, where k is an integer. Then $n = 2 * k - 3 = 2 * (k - 2) + 1$, i.e. n is odd. $\neg p$ is true.

  We have shown that $\neg q \rightarrow \neg p$; by contrapositive, we conclude $p \rightarrow q$.

- **Contradiction**:
  Hypothesis: We suppose that the proposition $p \rightarrow q$ is false, i.e. that p is true and q is false. If n is even, there exists an integer k such that $n = 2 * k$, Then, $n + 4 = 2 * k + 4 = 2 * (k + 2)$. Therefore, $n + 4$ is even; but the hypothesis states $n + 4$ is odd: we reach a contradiction. The hypothesis is wrong, therefore the statement $p \rightarrow q$ is true.

Exercise 2

There are at least two methods to show that $(A - B) - C = (A - C) - (B - C)$. I will use both a "direct" proof based on set theory identity and a proof based on logic, using a membership table.

- **Method 1: set identity**

$(A - C) - (B - C) =$	
$(x \in A \wedge \neg x \in C) \wedge \neg(x \in B \wedge \neg x \in C)$	Definition
$(x \in A \wedge \neg x \in C) \wedge (\neg x \in B \vee x \in C)$	De Morgan's law
$((x \in A \wedge \neg x \in C) \wedge \neg x \in B) \vee ((x \in A \wedge \neg x \in C) \wedge x \in C))$	Distributivity
$((x \in A \wedge \neg x \in C) \wedge \neg x \in B) \vee (x \in A \wedge (\neg x \in C \wedge x \in C))$	Associativity
$((x \in A \wedge \neg x \in C) \wedge \neg x \in B) \vee (x \in A \wedge F)$	Complement law
$((x \in A \wedge \neg x \in C) \wedge \neg x \in B) \vee F$	Absorption law
$(x \in A \wedge \neg x \in C) \wedge \neg x \in B$	Absorption law
$(x \in A \wedge \neg x \in B) \wedge \neg x \in C$	Associativity
$(A - B) - C$	Definition

- **Method 2: Membership table**

A	B	C	$A-B$	$A-C$	$B-C$	$(A-B)-C$	$(A-C)-(B-C)$
1	1	1	0	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	1	0	0	0	0
1	0	0	1	1	0	1	1
0	1	1	0	0	0	0	0
0	1	0	0	0	1	0	0
0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0

Since column 7 and 8 are equal, the two sets are equal

Exercise 3

Again, there are at least two methods to show that $A \oplus B = (A-B)-(B-A)$. I will use both a "direct" proof based on set theory identity and a proof based on logic, using a membership table.

- **Method 1: set identity**

$(A-B) \cup (B-A) =$	
$(x \in A \wedge \neg x \in B) \vee (x \in B \wedge \neg x \in A)$	Definition
$((x \in A \wedge \neg x \in B) \vee x \in B) \wedge ((x \in A \wedge \neg x \in B) \vee \neg x \in A)$	Distributivity
$((x \in A \vee x \in B) \wedge (\neg x \in B \vee x \in B)) \wedge ((x \in A \wedge \neg x \in B) \vee \neg x \in A)$	Distributivity
$((x \in A \vee x \in B) \wedge T) \wedge ((x \in A \wedge \neg x \in B) \vee \neg x \in A)$	Complement law
$(x \in A \vee x \in B) \wedge ((x \in A \wedge \neg x \in B) \vee \neg x \in A)$	Absorption law
$(x \in A \vee x \in B) \wedge ((x \in A \vee \neg x \in A) \wedge (\neg x \in B \vee \neg x \in A))$	Distributivity
$(x \in A \vee x \in B) \wedge (T \wedge (\neg x \in B \vee \neg x \in A))$	Complement law
$(x \in A \vee x \in B) \wedge (\neg x \in B \vee \neg x \in A)$	Absorption law
$(x \in A \vee x \in B) \wedge (\neg(x \in A \wedge x \in B))$	De Morgan's law
$A \oplus B$	Definition

- **Method 2: Membership table**

A	B	$A-B$	$B-A$	$(A-B) \cup (B-A)$	$A \oplus B$
1	1	0	0	0	0
1	0	1	0	1	1
0	1	0	1	1	1
0	0	0	0	0	0

Since column 5 and 6 are equal, the two sets are equal.

Exercise 4

All three problems can be solved using a membership table. Here I describe either alternative solution, using the definition of sets and their properties, or the solution based on the membership table (except for case a where I use both).

- **a)**

  - **Direct proof**

    From exercise 3, we know that:

    $$A \oplus B = (A - B) \cup (B - A)$$

    Applying this result by inverting A and B, we get:

    $$B \oplus A = (B - A) \cup (A - B) = (A - B) \cup (B - A)$$

    since $\cup$ is symmetric. Therefore $A \oplus B = B \oplus A$.

  - **Membership table:**

A	B	$A \oplus B$	$B \oplus A$
1	1	0	0
1	0	1	1
0	1	1	1
0	0	0	0

- **b)**

  I only use the membership table:

A	B	$A \oplus B$	$(A \oplus B) \oplus B$
1	1	0	1
1	0	1	1
0	1	1	0
0	0	0	0

- **c)**

  From exercise 3, we know that $A \oplus B = (A - B) \cup (B - A)$. Applying this result to $B = A$, we get $A \oplus A = \emptyset \cup \emptyset = \emptyset$.

  We need to show that: if $A \neq \emptyset$ then $A \oplus A \neq A$. This is an implication of the form $p \to q$ with

$p : A \neq \oslash$

and

$q : A \oplus A \neq A$

We show that the implication is true using a proof by contrapositive. Let us assume that $\neg q$ is true, i.e. that $A \oplus A = A$. As mentioned above, $A \oplus A = \oslash$. Therefore $A = \oslash$. We have shown $\neg p$ is true. Therefore $\neg q \to \neg p$ is true, and $p \to q$ is true.

Exercise 5

a: Let $A = \{1, 2, 3\}$, $B = \{1, 4\}$, $C = \{3, 4\}$. Then $A \cup B = B \cup C = \{1, 2, 3, 4\}$ and $A \neq B$.
b. Let $A = \{1, 2, 3\}$, $B = \{2, 4\}$, $C = \{2, 5\}$. Then $A \cap B = B \cap C = \{2\}$ and $A \neq B$.

Exercise 6

From the inclusion-exclusion principle, we know that:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Let us consider the three sets A, B and C. We observe that:

$$|A \cup B \cup C| = |(A \cup B) \cup C|$$

Then:

$$\begin{aligned}
|A \cup B \cup C| &= |A \cup B| + |C| - |(A \cup B) \cap C| \\
&= |A| + |B| + |C| - |A \cap B| - |(A \cup B) \cap C| \\
&= |A| + |B| + |C| - |A \cap B| - |(A \cap C) \cup (B \cap C)| \\
&= |A| + |B| + |C| - |A \cap B| - (|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|) \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap C) \cap (B \cap C)| \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\end{aligned}$$

Exercise 7

We want to show that

$$|\bar{A} \cap \bar{B}| = |U| - |A| - |B| + |A \cap B|$$

Let us define $C = A \cup B$.
A complement law tells us that $C \cup \bar{C} = U$ where U is the domain. Also, $C \cap \bar{C} = \oslash$. Using the inclusion-exclusion principle (see exercise 6), we find:

$$|U| = |C| + |\bar{C}|$$

We know that $|C| = |A \cup B| = |A| + |B| - |A \cap B|$, and, based on DeMorgan's law, $\bar{C} = \bar{A} \cap \bar{B}$. Replacing in the equation above, after rearrangement, we obtain:

$$|\bar{A} \cap \bar{B}| = |U| - |A| - |B| + |A \cap B|$$

Exercise 8

a: $\bigcup_{i=1}^{n} A_i = \{..., -2, -1, 0, 1, ..., n\} = A_n$
b: $\bigcap_{i=1}^{n} A_i = \{..., -2, -1, 0, 1\} = A_1$

Exercise 9

We want to show that if $A \bigcup B = B$, then $A \bigcap B = A$.

This is a proof of an implication. We use a direct proof.

Let A and B be two sets in a domain D. To show that $A \bigcap B = A$, we show that $A \bigcap B \subset A$ and $A \subset A \bigcap B$

a) $A \bigcap B \subset A$.

   Let x be an element of $A \bigcap B$. Then $x \in A$ and $x \in b$, and a fortiori $x \in A$.

b) $A \subset A \bigcap B$

   Let x be an element of A. Since $A \subset A \bigcup B$, $x \in A \bigcup B$. The premise is that $A \bigcup B = B$. Therefore $x \in B$. Since x is in A and in B, $x \in A \bigcap B$.

This concludes the proof.

Exercise 10

We want to show that if $A \bigcap B = A$, then $B \bigcap \left(\overline{B \bigcap \overline{A}}\right) = A$.

This is a proof of an implication. We use a direct proof.

We simplify $B \bigcap \left(B \bar{\bigcap} \bar{A}\right)$:

$$\begin{array}{rcl|c} B \bigcap \left(\overline{B \bigcap \overline{A}}\right) & = & B \bigcap \left(\overline{B} \bigcup \overline{\overline{A}}\right) & \text{De Morgan's law} \\ & = & B \bigcap (\overline{B} \bigcup A) & \text{Complementation law} \\ & = & (B \bigcap \overline{B}) \bigcup (B \bigcap A) & \text{Distributivity law} \\ & = & \emptyset \bigcup (B \bigcap A) & \text{Complement set 1} \\ & = & B \bigcap A & \text{Absorption law 1} \end{array}$$

Therefore, $B \bigcap \left(\overline{B \bigcap \overline{A}}\right) = B \bigcap A = A \bigcap B$. According to the premise, $A \bigcap B = A$. Therefore $B \bigcap \left(\overline{B \bigcap \overline{A}}\right) = A$. This concludes the proof.

Extra Credit

a) Let A and B be two sets. We want to show that $\mathcal{P}(A) \bigcap \mathcal{P}(B) = \mathcal{P}(A \bigcap B)$.

   Let us define $LHS = \mathcal{P}(A) \bigcap \mathcal{P}(B)$ and $RHS = \mathcal{P}(A \bigcap B)$. We want to show $LHS = RHS$. We will show that $LHS \subset RHS$ and $RHS \subset LHS$.

   i) Let $S \in LHS$. By definition of LHS, $S \in \mathcal{P}(A)$ and $S \in \mathcal{P}(A)$. Since $S \in \mathcal{P}(A)$, $S \subset A$. Similarly, since $S \in \mathcal{P}(B)$, $S \subset B$. Therefore S is a subset of $A \bigcap B$, which is equivalent to saying that $S \in \mathcal{P}(A \bigcap B)$.

ii) Let $S \in RHS$. By definition of RHS, $S \in \mathcal{P}(A \bigcap B)$, therefore $S \subset A \bigcap B$. By definition of $A \bigcap B$, we have that $S \subset A$ and $S \subset B$, i.e. $S \in \mathcal{P}(A)$ and $S \in \mathcal{P}(B)$, i.e. $S \in \mathcal{P}(A) \bigcap \mathcal{P}(B)$.

We conclude that $\mathcal{P}(A) \bigcap \mathcal{P}(B) = \mathcal{P}(A \bigcap B)$.

b) We want to show that there exists two sets A and B such that $\mathcal{P}(A) \bigcup \mathcal{P}(B) \neq \mathcal{P}(A \bigcup B)$.

Let $A = \{a, b\}$ and $B = \{c, d\}$. then:

$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

and

$\mathcal{P}(B) = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$

Therefore,

$\mathcal{P}(A) \bigcup \mathcal{P}(B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}\}$

As $A \bigcup B = \{a, b, c, d\}$ we have,

$\mathcal{P}(A \bigcup \mathcal{P}(B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}\{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$

Therefore $\mathcal{P}(A) \bigcup \mathcal{P}(B) \neq \mathcal{P}(A \bigcup B)$.