Homework 5 Solutions

ECS 20 (Fall 16)

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October 18, 2016

Exercise 1

a: 2
b: 3
c: -4
d: -3
e: 7
f: -6
g: 1
h: 2

Exercise 2

a) Show that the statement
"If x is a real number such that $x^2 + 2 = 0$, then $x^4 = -5$ "
is true.

Let P be the statement. P is an implication of the form $p \rightarrow q$ with p defined as "x is a real number with $x^2 + 2 = 0$" and q defined as "$x^4 = -5$". As p is false, the proposition P is always true.

b) If x and y are real numbers such that $x < y$, show that there exists a real number z with $x < z < y$.
This is an existence proof: we only need to find one example. Let us define $z = \frac{x+y}{2}$. We show that $x < z$ and $z < y$.
$z - x = \frac{x+y}{2} - x = \frac{y-x}{2} > 0$ as $x < y$.
Similarly,
$y - z = y - \frac{x+y}{2} = \frac{y-x}{2} > 0$ as $x < y$.
Therefore $x < z < y$: we found one real number z that satisfies $x < z < y$: this concludes the proof.

Exercise 3

Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Let us write $x = n + \epsilon$, where n is an integer and ϵ is a real number and $0 \leq \epsilon < 1$. n is the largest integer that is smaller than x; by definition, $n = \lfloor x \rfloor$. We use a proof by case (similar to the proof used in class for $\lfloor 2x \rfloor$):

a) : If $0 \leq \epsilon < 1/3$, then $0 \leq 3\epsilon < 1$, $0 < \epsilon + 1/3 < 1$ and $0 < \epsilon + 2/3 < 1$. Therefore,

$$\lfloor 3x \rfloor = \lfloor 3n + 3\epsilon \rfloor = 3n$$

and

$$\begin{aligned} \lfloor x \rfloor + \lfloor x + 1/3 \rfloor + \lfloor x + 2/3 \rfloor &= \lfloor n + \epsilon \rfloor + \lfloor n + \epsilon + 1/3 \rfloor + \lfloor n + \epsilon + 2/3 \rfloor \\ &= n + n + n \\ &= 3n \end{aligned}$$

b) If $1/3 \leq \epsilon < 2/3$, then $1 \leq 3\epsilon < 2$, $0 < \epsilon + 1/3 < 1$ and $1 \leq \epsilon + 2/3 < 2$. Therefore,

$$\lfloor 3x \rfloor = \lfloor 3n + 3\epsilon \rfloor = 3n + 1$$

and

$$\begin{aligned} \lfloor x \rfloor + \lfloor x + 1/3 \rfloor + \lfloor x + 2/3 \rfloor &= \lfloor n + \epsilon \rfloor + \lfloor n + \epsilon + 1/3 \rfloor + \lfloor n + \epsilon + 2/3 \rfloor \\ &= n + n + n + 1 \\ &= 3n + 1 \end{aligned}$$

c) If $2/3 \leq \epsilon < 1$, then $2 \leq 3\epsilon < 3$, $1 < \epsilon + 1/3 < 2$ and $1 \leq \epsilon + 2/3 < 2$. Therefore,

$$\lfloor 3x \rfloor = \lfloor 3n + 3\epsilon \rfloor = 3n + 2$$

$$\begin{aligned} \lfloor x \rfloor + \lfloor x + 1/3 \rfloor + \lfloor x + 2/3 \rfloor &= \lfloor n + \epsilon \rfloor + \lfloor n + \epsilon + 1/3 \rfloor + \lfloor n + \epsilon + 2/3 \rfloor \\ &= n + n + 1 + n + 1 \\ &= 3n + 2 \end{aligned}$$

Based on the method of proof by case, we conclude that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + 1/3 \rfloor + \lfloor x + 2/3 \rfloor$ is true for all x.

Exercise 4

Show that for all real number x and all strictly positive integer n, $\left\lfloor \frac{\lfloor nx \rfloor}{n} \right\rfloor = \lfloor x \rfloor$
Let us define $k = \lfloor nx \rfloor$ and $m = \lfloor x \rfloor$. By definition of floor, we have the two properties:
$k \leq nx < k+1$
and
$m \leq x < m+1$
Let us multiply the second inequalities by n:
$nm \leq nx < n(m+1)$
We notice that:
$k \leq nx$ and $nx < n(m+1)$; therefore $k < n(m+1)$.
$k \leq nx$ and $nm \leq nx$. Therefore k and nm are two integers smaller than nx. By definition of floor, k is the largest integer smaller that nx. Therefore $nm \leq k$.

Combining those two inequalities, we get $nm \leq k < n(m+1)$. After division by n, $m < \frac{k}{n} < m+1$. Therefore m is the floor of $\frac{k}{n}$. Replacing m and k by their values, we get:

$$m = \lfloor x \rfloor = \left\lfloor \frac{k}{n} \right\rfloor = \left\lfloor \frac{\lfloor nx \rfloor}{n} \right\rfloor$$

The property is therefore true.

Extra Credit

This is a generalization of exercise 3:

Let x be a real number and N an integer greater or equal to 3. Show that $\lfloor Nx \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{N} \rfloor + \lfloor x + \frac{2}{N} \rfloor + \ldots + \lfloor x + \frac{N-1}{N} \rfloor$.

We could use a proof by case that generalizes the solution described for exercise 1, using N case; there is however a faster and maybe more elegant solution.

Let us define:

$f(x) = \lfloor Nx \rfloor - \lfloor x \rfloor - \lfloor x + \frac{1}{N} \rfloor - \lfloor x + \frac{2}{N} \rfloor - \ldots - \lfloor x + \frac{N-1}{N} \rfloor$

We show first that $f(x)$ is periodic, with $\frac{1}{N}$ being one period. For this, we need to show that: $\forall x \in \mathbb{R}, \quad f\left(x + \frac{1}{N}\right) = f(x)$

Let x be a real number. Notice that:

$$\begin{aligned} f\left(x + \frac{1}{N}\right) &= \lfloor N(x + \frac{1}{N}) \rfloor - \lfloor x + \frac{1}{N} \rfloor - \lfloor x + \frac{2}{N} \rfloor - \ldots - \lfloor x + \frac{1}{N} + \frac{N-2}{N} \rfloor - \lfloor x + \frac{1}{N} + \frac{N-1}{N} \rfloor \\ &= \lfloor Nx + 1 \rfloor - \lfloor x + \frac{1}{N} \rfloor - \ldots - \lfloor x + \frac{N-1}{N} \rfloor - \lfloor x + 1 \rfloor \\ &= \lfloor Nx \rfloor + 1 - \lfloor x + \frac{1}{N} \rfloor - \ldots - \lfloor x + \frac{N-1}{N} \rfloor - \lfloor x \rfloor - 1 \\ &= f(x) \end{aligned}$$

Since this is true with no conditions on x, it is true for all x, and therefore f is periodic, with $1/N$ being one period.

A periodic function needs to be defined only on one period, here in the interval $\left[0, \frac{1}{N}\right)$. Let x be in this interval. Then:

$$
\begin{aligned}
&0 \leq x < \frac{1}{N} < 1 \\
&0 \leq x + \frac{1}{N} < \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < 1 \\
&\dots \\
&0 \leq x + \frac{N-1}{N} < \frac{1}{N} + \frac{N-1}{N} = \frac{N}{N} = 1 \\
&0 \leq Nx < N\frac{1}{N} = 1
\end{aligned}
$$

Therefore $f(x) = 0$.

Since $f(x) = 0$ on one of its period, we have $f(x) = 0 \quad \forall x \in \mathbb{R}$. Therefore:

$\lfloor Nx \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{N} \rfloor + \lfloor x + \frac{2}{N} \rfloor + \ldots + \lfloor x + \frac{N-1}{N} \rfloor$