# Homework 6 Solutions

ECS 20 (Fall 2016)

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## Exercise 1

a) Let us show that $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$ is true for all real number $x$. Let us define $y = \lfloor x \rfloor$; as $y$ is an integer, the ceiling value of $y$ is equal to $y$, i.e. $\lceil y \rceil = y$. Replacing $y$ by its value, we get, $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$.

b) Let x=1.6 and y=2.0. $\lfloor xy \rfloor = \lfloor 3.2 \rfloor = 3$. However, $\lfloor x \rfloor \times \lfloor y \rfloor = 1 \times 2 = 2$. Therefore, we have found a counter example to the proposition$\lfloor x \times y \rfloor = \lfloor x \rfloor \times \lfloor y \rfloor$ for all $x \in \mathbb{R}$.

c) The property does not hold for all positive real numbers. Let us consider $x = 0.25$. $\lfloor \sqrt{x} \rfloor = \lfloor 0.5 \rfloor = 0$ and $\lfloor \sqrt{\lceil 0.25 \rceil} \rfloor = \lfloor \sqrt{1} \rfloor = \lfloor 1 \rfloor = 1$.

## Exercise 2

Show that $x^3$ is $\mathcal{O}(x^4)$ but that $x^4$ is not $\mathcal{O}(x^3)$.

a) Let us show that $x^3$ is $\mathcal{O}(x^4)$

Let us assume that $1 < x$. Since $x > 0$, we can multiply this inequality by $x$: $x < x^2$, again: $x^2 < x^3$ and finally $x^3 < x^4$.

We have shown that there exists $k$ ($k = 1$), and there exists $C$ ($C = 1$), such that if $x > k$, then $x^3 < Cx^4$: we can conclude that $x^3$ is $\mathcal{O}(x^4)$.

b) Let us show that $x^4$ is not $\mathcal{O}(x^3)$.

We use a proof by contradiction: let us suppose that the proposition is true, i.e. that $x^4$ is $\mathcal{O}(x^3)$. By definition of $\mathcal{O}$, this means that:

$\exists k \in \mathbb{R}, \exists C \in \mathbb{R}$ if $x > k$ then $|x^4| < C|x^3|$.

Let $D = \max\{2, k, C\}$. Therefore $D > 0$, $D \geq k$, and $D \geq C$.

Let $x$ be a real number with $x > D$. Since $D \geq k$, we have $x^4 < Dx^3$. Since $x > 0$, we can divide this inequality by $x^3$: we get $x < D$. But we supposed that $x > D$: we have reached a contradiction. Therefore, the hypothesis, $x^4$ is $\mathcal{O}(x^3)$, is false. We can conclude that $x^4$ is not $\mathcal{O}(x^3)$.

## Exercise 3

a) Show that $2x - 9$ is $\Theta(x)$.

One option is to prove that $2x - 9$ is both $\mathcal{O}(x)$ and $\Omega(x)$. In this simple case however, we directly "squeeze" $2x - 9$ between two functions that are of order $x$. First, lest us notice that $\forall x \in \mathbb{R}, 2x - 9 < 2x$.

Second, we note that if $x > 9$, then $x - 9 > 0$ and therefore $x + x - 9 > x$, i.e. $2x - 9 > x$.

Summarizing: for $x > 9$, $x < 2x - 9 < 2x$. Therefore $2x - 9$ is $\Theta(x)$.

b) Show that $3x^2 + x - 5$ is $\Theta(x^2)$.

Again, one option is to prove that $3x^2 + x - 5$ is both $\mathcal{O}(x^2)$ and $\Omega(x^2)$. In this simple case however, we directly "squeeze" $3x^2 + x - 5$ between two functions that are of order $x^2$.

We note first that when $x > 5$, then $x - 5 > 0$, and therefore $3x^2 + x - 5 > 3x^2$.

Second, we note that when $1 < x$, $x < x^2$, and therefore $x - 5 < x^2 - 5 < x^2$. This leads to $3x^2 + x - 5 < 4x^2$ when $x > 1$.

Summarizing: for $x > 5$, $3x^2 < 3x^2 + x - 5 < 4x^2$. Therefore $3x^2 + x - 5$ is $\Theta(x^2)$.

c) Show that $\lfloor x + \frac{2}{3} \rfloor$ is $\Theta(x)$.

Again, we will "squeeze" $\lfloor x + \frac{2}{3} \rfloor$ between two functions that are or order $x$.

By definition of the function floor, $\lfloor x + \frac{2}{3} \rfloor \leq x + \frac{2}{3}$. If $\frac{2}{3} < x$, this leads to $\lfloor x + \frac{2}{3} \rfloor < 2x$.

Similarly, $x + \frac{2}{3} < \lfloor x + \frac{2}{3} \rfloor + 1$, which we rewrite as $x - \frac{1}{3} < \lfloor x + \frac{2}{3} \rfloor$.

If $x > 1$, then $\frac{x}{3} > \frac{1}{3}$; multiplying by $-1$, $-\frac{x}{3} < -\frac{1}{3}$, and adding $x$, we get $x - \frac{x}{3} < x - \frac{1}{3}$, namely $\frac{2x}{3} < x - \frac{1}{3}$, therefore $\frac{2x}{3} < \lfloor x + \frac{2}{3} \rfloor$.

Summarizing: for $x > 1$, $\frac{2x}{3} < \lfloor x + \frac{2}{3} \rfloor < 2x$. Therefore $\lfloor x + \frac{2}{3} \rfloor$ is $\Theta(x)$.

d) Show that $\log_{10}(x)$ is $\Theta \log_2(x)$.

Notice first that $\log_{10}(x) = \log_{10}(2) \times \log_2(x)$. Since $\log_{10}(x)$ and $\log_2(x)$ only differ by a (positive) constant, there are of the same order. Hence $\log_{10}(x)$ is $\Theta(\log_2(x))$.

## Exercise 4

Let $a$ and $b$ be two integers. We want to prove that

If $a^2 - b^2 + 2ab$ is odd, then $a - b$ is odd

using a proof by contradiction.

Let

$p : a^2 - b^2 + 2ab$ is odd

and

$q : a - b$ is odd

Proof by contradiction: we suppose that $p \to q$ is false, i.e. that $p$ is true AND $q$ is false.

Since $q$ is false, $a - b$ is even: there exists an integer $k$ such that $a - b = 2k$. Then

$a^2 - b^2 + 2ab = (a - b)(a + b) + 2ab = 2k(a + b) + 2ab = 2[k(a + b) + ab]$, i.e. $a^2 - b^2 + 2ab$ is even. However, we have supposed $p$ is true, namely that $a^2 - b^2 + 2ab$ is odd. We have reached a contradiction.

Therefore $p \to q$ is true.

## Exercise 5

Use a proof by contradiction to show that:

$p$ : There exists a strictly positive real number $r$ such that, for all real number $x$, if $x - \lfloor x \rfloor < r$ then $\lfloor 3x \rfloor = 3x$.

Assumption: $p$ is false. This means that:

For all strictly positive real numbers $r$, there exists a real number $x$ such that $x - \lfloor x \rfloor < r$ AND $\lfloor 3x \rfloor \neq 3x$.

Let us assume that we have found such a real number $x$. Since $x - \lfloor x \rfloor < r$ for all strictly positive number $r$, we have that $x - \lfloor x \rfloor \leq 0$. We know also that $x - \lfloor x \rfloor \geq 0$ (by definition of floor). Therefore $x = \lfloor x \rfloor$, i.e. $x$ is an integer.

Since $x$ is an integer, $3x$ is an integer, and therefore $\lfloor 3x \rfloor = 3x$. However, we had assumed that $\lfloor 3x \rfloor \neq 3x$. We have reached a contradiction and therefore such a real number $x$ does not exist; the proposition $\neg p$ is false, therefore $p$ is true!

## Extra Credit

a) List of divisors of 6, not including 6: $F_6 = 1, 2, 3$. Sum($F_6$) = 1+2+3 = 6.
List of factors of of 28, not including 28: $F_{28} = 1, 2, 4, 7, 14$. Sum($F_{28}$)=1+2+4+7+14 = 28.

Since 6 and 28 are sums of their respective factors (excluding themselves) , we have shown that 6 and 28 are perfect numbers.

b) Suppose that $(2^p - 1)$ is prime, and let $n = 2^{p-1} \times (2^p - 1)$. The list of divisors of $n$ is given by :

$$\begin{aligned}
D_n \quad = \quad & \{1, (2^p - 1), \\
& 2, 2(2^p - 1), \\
& 2^2, 2^2(2^p - 1), \\
& 2^3, 2^3(2^p - 1), \\
& \ldots \\
& 2^{p-1}, 2^{p-1}(2^p - 1)\}
\end{aligned}$$

We need to exclude $n$ from this list, who becomes:

$$\begin{aligned}
D_n^* \quad = \quad & \{1, (2^p - 1), \\
& 2, 2(2^p - 1), \\
& 2^2, 2^2(2^p - 1), \\
& 2^3, 2^3(2^p - 1), \\
& \ldots \\
& 2^{p-1}\}
\end{aligned}$$

Sum of all the divisors in $D_n^*$, $S_n$, is given by:

$$\begin{aligned} S_n &= (1 + 2 + 2^2 + \ldots + 2^{p-1}) + (1 + 2 + 2^2 + \ldots + 2^{p-2}) \times (2^p - 1) \\ &= 2^p - 1 + (2^{p-1} - 1) \times (2^p - 1) \\ &= (2^p - 1) \times (1 + 2^{p-1} - 1) \\ &= (2^p - 1) \times 2^{p-1} \qquad (1) \\ &= n \end{aligned}$$

Therefore $n$ is a perfect number.