Set Theory

I. Sets

1.1 Definition A set is an unordered collection of object.

Examples

- Set S of all odd numbers between 0 and 10

$$S = \{1, 3, 5, 7, 9\}$$

- Days of the week

$$W = \{M, T, W, Th, F, S, S\}$$

- Sets of numbers:

$$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$$

1.2. How to describe a set

A set is well-defined if there is a clear rule that defines membership.

Let us consider the set S of all integers between 2 and 40.

There are two methods to describe S symbollically:

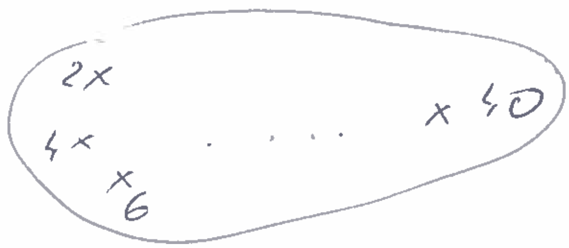
- The roster notation is a complete, or implied listing of all elements of S

$$S = \{2, 4, 6, 8, \ldots, 40\}$$

- The set-builder notation is a symbolic representation used when the roster notation is cumbersome

$$S = \{x \mid x \in \mathbb{N} \text{ and } 2 \leq x \leq 40 \text{ and } x \text{ is even}\}$$

There is an additional method that is useful to get intuition about set: the Venn diagram



1.3 Paradoxes

The objects, or elements of a set do not have to be numbers. We can even build sets of sets. This leads to some paradoxes:

The barber's paradox Suppose there is a town with one barber, and all men keep clean shaven. Let S be the set of men that are shaved by the barber. Can we define S with: "The barber shaves all, and only those men who do not shave themselves".

Does the barber belong to S? $\rightarrow$ paradox

Russell's paradox

let M be the set of all sets that do not contain themselves as members.

$$M = \{ A \mid A \notin A \}$$

M seems to be well defined, i.e. there is an explicit rule that defines membership.

However .... does M belongs to M?

if $M \in M$, then $M \notin M$
if $M \notin M$, then M should belong to M
] paradox!

We will leave this to logisticians.

1.4 Terminology

Symbol	Meaning	Example	Reads as
$\in$	element of	$x \in A$	x is an element of A
$\subset$	subset	$A \subset B$	A is a subset of B
$\cup$	union	$A \cup B$	A union B
$\cap$	intersection	$A \cap B$	A intersect B
$-$	difference	$A - B$ or $A \setminus B$	difference of A and B
$\overline{}$	complement	$\overline{A}$	complement of A
$\lvert\ \rvert$	cardinality	$\lvert A \rvert$	cardinal, or number of elements of A (A finite)

1.5 Definitions

A) **Subset**: A is a subset of B if and only if every element of A is an element of B

$$\forall x \in A, \quad x \in B$$

Notes: a) There is one special subset, the empty set, noted $\emptyset$ that belongs to all sets.

b) A is a subset of A: $A \subset A$

B) **Union**: The union of the sets A and B, denoted $A \cup B$, is the set that contains those elements that are either in A, or in B, ~~are~~ in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

C) **Intersection**: The intersection of the sets A and B, denoted $A \cap B$, is the set that contains those elements that are both in A and B

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

D) **Difference**: The difference of the sets A and B, denoted $A \setminus B$ or $A - B$, is the set that contains those elements that are in A, but not in B

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

E) Complement

Let $A = \{1, 2, 3\}$. What would be the complement of A? Elements that do not belong to A? Then you would belong to this complement!

. We need a robust definition.

First, we need to define the domain, a universal set, or universe, D.

Then $\overline{A} = \{x \mid x \in D \wedge x \notin A\}$

F) Cardinality

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If a set S is finite, the cardinality of S, denoted $|S|$, is the number of elements of S

$|\emptyset| = 0$

Addition principle: if A and B are two sets that are disjoints, i.e $A \cap B = \emptyset$ then

$$|A \cup B| = |A| + |B|$$

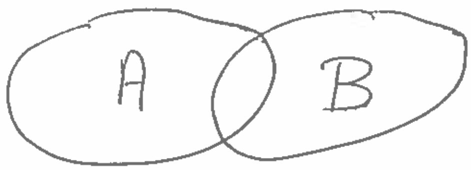
Proof: obvious!

Inclusion-exclusion principle:

Let A and B be two sets in the same universe U. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof:



1) $A \cup B = A \cup (B - A)$

$$A \cap (B - A) = \{x \in U \,/\, x \in A \wedge (x \in B - A)\}$$
$$= \{x \in U \,/\, x \in A \wedge (x \in B \wedge x \notin A)\}$$
$$= \emptyset$$

Therefore

$$|A \cup B| = |A| + |B - A| \qquad (1)$$

2) $B = (B - A) \cup (A \cap B)$

$$(B - A) \cap (A \cap B) = \{x \in U \,/\, (x \in B \wedge x \notin A) \wedge (x \in A \wedge x \in B)\}$$
$$= \emptyset$$

Therefore: $|B| = |B - A| + |A \cap B| \qquad (2)$

(1) - (2) :

$$|A \cup B| = |A| + |B| + |A \cap B|$$

~~2.~~ Set identities

A, B, and C sets in a domain, a universe $\mathcal{U}$

Identity laws: $A \cup \emptyset = A$

$A \cap \mathcal{U} = A$

Denomination laws: $A \cup \mathcal{U} = \mathcal{U}$

$A \cap \emptyset = \emptyset$

Idempotent $A \cup A = A$

$A \cap A = A$

Double Complement $\overline{\overline{A}} = A$

Distributive laws: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Magan's laws $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

Complement laws $A \cup \overline{A} = \mathcal{U}$

$A \cap \overline{A} = \emptyset$

Two examples of proofs

1) De Morgan's law: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

We show that $\overline{A \cap B} \subset \overline{A} \cup \overline{B}$ (1)

and $\overline{A} \cup \overline{B} \subset \overline{A \cap B}$ (2)

(1). Let $x \in \overline{A \cap B}$: $x \notin A \cap B$

$\neg(x \in A \wedge x \in B)$	Definition of complement and intersection
$\neg(x \in A) \vee \neg(x \in B)$	De Morgan's law
$x \in \overline{A} \vee x \in \overline{B}$	definition of complement
$x \in \overline{A} \cup \overline{B}$	definition of union

(2) Let $x \in \overline{A} \cup \overline{B}$

$x \in \overline{A} \vee x \in \overline{B}$	Definition of union
$\neg x \in A \vee \neg x \in B$	Definition of complement
$\neg(x \in A \wedge x \in B)$	de Morgan's law
$\neg(x \in A \cap B)$	definition of intersection
$x \in \overline{A \cap B}$	definition of complement.

Alternate proof using table of truth:

Let A be a set in a universe U. We write "1" for x if $x \in A$, and "0" if $x \in U - A$

We can then generate the table of "truth". (9)

A	B	$\overline{A}$	$\overline{B}$	$A \cap B$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	0	1	1

equivalent.

Therefore $\overline{A \cap B} = \overline{A} \cup \overline{B}$

2) Distributivity

Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

We show that ① $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$

② $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$

① Let $x \in A \cup (B \cap C)$

$x \in A \vee x \in (B \cap C)$	Definition of union
$x \in A \vee (x \in B \wedge x \in C)$	Definition of intersection
$(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$	Distributivity
$x \in (A \cup B) \wedge x \in (A \cup C)$	Definition of union
$x \in (A \cup B) \cap (A \cup C)$	Definition of intersection

② Let $x \in (A \cup B) \cap (A \cup C)$

$x \in (A \cup B) \wedge x \in (A \cup C)$	Definition of intersection
$(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$	Definition of union
$x \in A \vee (x \in B \wedge x \in C)$	Distributivity
$x \in A \vee x \in (B \cap C)$	Definition of intersection
$x \in A \cup (B \cap C)$	Definition of union

Proof using truth table:

A, B, C in a universe U

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

equivalent

Therefore $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Example of application

Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

$$\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{B \cap C} \quad \text{De Morgan's law}$$

$$= \overline{A} \cap (\overline{B} \cup \overline{C}) \quad \text{De Morgan's law}$$

$$= \overline{A} \cap (\overline{C} \cup \overline{B}) \quad \text{Commutativity}$$

$$= (\overline{C} \cup \overline{B}) \cap \overline{A} \quad \text{Commutativity}$$

3. Generalized sets

3.1 Generalized union

The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

$$A_1 \cup A_2 \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

3.2 Generalized intersection

The intersection of a collection of sets is the set that contains those elements that are members of all sets in the collection

$$A_1 \cap A_2 \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

Symbolic representation of generalized set:

$$\bigcup_{i=1}^{N} A_i = \{x \,/\, \exists i \in [1, \dots N],\ x \in A_i\}$$

$$\bigcap_{i=1}^{N} A_i = \{x \,/\, \forall i \in [1, \dots N],\ x \in A_i\}$$

Negation

$$x \notin \bigcup_{i=1}^{N} A_i \quad \text{means} \quad \forall i \in [1, \dots N]\ x \notin A_i$$

$$x \notin \bigcap_{i=1}^{N} A_i \quad \text{means} \quad \exists\, i \in [1, \dots N]\ x \notin A_i$$

3.3. The power set

Givent a set S, the power set of S, $\mathcal{P}(S)$ is the set of all subset of S.

Example: What is the power set of $S = \{a, b, c\}$

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\} \{a,b,c\}\}$$

3.4 Cartesian product

Let A and B be sets. The Cartesian product of A and B, denoted $A \times B$, is the set of all ordered pair (a,b), where $a \in A$ and $b \in B$

$$A \times B = \{ (a,b) \mid a \in A \wedge b \in B \}$$

4. Computer representation of sets

There are various ways to represent sets using a computer. One method is to store the elements in an unordered list, or array. Operations on such lists however can be time consuming.

Bit string representation of a set

Let D be the finite domain under consideration, whose element are $D = \{a_1, \ldots a_N\}$.

A subset A of D is represented by the bit string of length N, such that the i-th bit in this string is 1 if a_i belongs to A, and 0 otherwise.

Example:

Let $D = \{a, b, c, d, e, f, g, h, i\}$

Let A be the subset of the vowels contained in D. The bit string representation of A is:

$$S_A = 1\,0\,0\,0\,1\,0\,0\,0\,1$$

Let B be the subset of the consonants contained in D. $B = \overline{A}$ and

$$S_B = 0\,1\,1\,1\,0\,1\,1\,1\,0$$