Logic
(mathematical language)

1) Propositions

Definition: A proposition, or statement is a declarative sentence that is either true (T) or false (F)

Examples:

Sentence	Proposition or not	Truth value.
$1+1=2$	Yes	True
$1+4=3$	Yes	False
Today is Sunday	Yes	Don't know
$x+3=2$	No	
I am lying now		

Notation P a proposition, T or F

Table of truth:

P
T
F

Compound proposition

1) **Negation**: Let p be a proposition. The sentence "It is not the case that p" is another proposition, called the negation of p, denoted $\neg p$ $(\sim p)$. It is read "not p".

Truth table

p	$\neg p$
T	F
F	T

2) **Conjunction**: The conjunction of two propositions p and q is the compound proposition that we write $p \wedge q$, and read "p and q", which is only true if p is true and q is true.

Table of truth:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction: The disjunction of 2 propositions p and q is the compound proposition written as $p \vee q$, and read "p or q" that is true if either p is true, or q is true, or both are true.

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T	T	T	F
T	F	F	T	T
F	T	F	T	T
F	F	F	F	F

Exclusive or: $p \oplus q$

Logical equivalence: Two propositions p and q are logically equivalent if they always have the same truth values. Notation: we write $p \Leftrightarrow q$

Statement For 2 propositions p and q

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

Truth table:

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Show that $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$(\neg p \wedge \neg q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Table of truth:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T	F	T	T	F
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	F	F	F

$$\boxed{p \oplus q \iff (p \vee q) \wedge (\neg (p \wedge q))}$$

De Magan's laws:

$$\neg (p \wedge q) \iff \neg p \vee \neg q$$

$$\neg (p \vee q) \iff \neg p \wedge \neg q$$

$$p \wedge \neg p \iff F$$

A contradiction is a statement that is always false.

$$p \vee \neg p \iff T$$

A tautology is a statement that is always true.

Show that $(p \vee q) \vee (\neg p \wedge \neg q)$ is a tautology.

Proof by logical equivalence:

$$(p \vee q) \vee (\neg p \wedge \neg q) \iff (p \vee q) \vee (\neg(p \vee q))$$

De Magan's law

Let us define $A = p \vee q$

Then

$$(p \vee q) \vee (\neg p \wedge \neg q) \iff A \vee \neg A$$
$$\iff T$$

$$a * (b + c) = a * b + a * c$$

$\cancel{p \wedge (p \vee q}$

$$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$$

You arrive on Smullyan's island and you meet Bill and Sally.

Bill: "The two of us are knights".

Sally: " Bill is a knave

Bill	Sally	Statement/Bill	Statement/Sally	
~~Knight~~	~~Knight~~	~~T~~	~~F~~	
~~Knight~~	~~Knave~~	~~F~~	~~F~~	
Knave	Knight	F	T	✓
~~Knave~~	~~Knave~~	~~F~~	~~T~~	

① : Sally is a knight that would lie.

② Bill is a knight that would lie.

④ Sally is a knave that would tell the truth.

Bill: I am a knave or Sally is a knight.

Sally: I am a knight.

Bill	Sally	P_B	P_S	
Knight	Knight	T	T	✓
~~Knight~~	~~Knave~~	~~F~~	~~F~~	
~~Knave~~	~~Knight~~	~~T~~	~~T~~	
~~Knave~~	~~Knave~~	~~T~~	~~F~~	

Bill: We are the same kind

Sally: We are of different kind.

Bill	Sally	P_B	P_S	
~~Knight~~	~~Knight~~	~~T~~	~~F~~	
~~Knight~~	~~Knave~~	~~F~~	~~T~~	
Knave	Knight	F	T	✓
~~Knave~~	~~Knave~~	~~T~~	~~F~~	