Geometric Objects and Transformations

How to represent basic geometric types, such as points and vectors?

How to convert between various represenations?

(through a linear transformation!)

How to establish a method for dealing with geometric problems independent of coordinate systems?

Homogeneous coordinates

Affine transformations

OpenGL transformation matrices
(in discussion session)

Mathematical Spaces

Linear Vector Spaces

- Scalars and vectors

- Examples of vector spaces:
  1. geometric vectors
  2. n−tuples of real numbers
  3. scalar−vector multiplication and vector−vector addition.
- The greatest number of linearly independent vectors that we can find in a space gives the **dimension** of the space
- **Bases**

Affine Spaces

- Add points −> location
- point−to−point subtraction yields a vector
- vector−point multiplication yields a point
- **Frames**

Euclidean Spaces

- Add measure of size or distance
- Inner product

Fundamental geometric objects:

point − a location in space

vector − a directed line seqment

Coordindate systems (frames):

a point can be represented unambiguously with a fixed reference point (the origin)

a vector can be uniquely defined as:

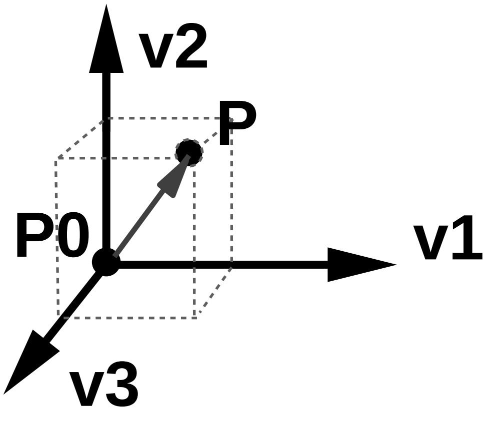
$$w = \alpha1\ v1 + \alpha2\ v2 + \alpha3\ v3$$

$$a^T = [\ \alpha1\ \alpha2\ \alpha3\ ]$$

$$w = a^T \begin{bmatrix} \mathbf{v1} \\ \mathbf{v2} \\ \mathbf{v3} \end{bmatrix}$$

then a point in a coordinate system defined by P0 (the origin) and a basis vector (v1, v2, v3) can be written as:

$$P = P0 + \ u\,v1 + v\,v2 + w\,v3$$



Changes of Coordinate Systems

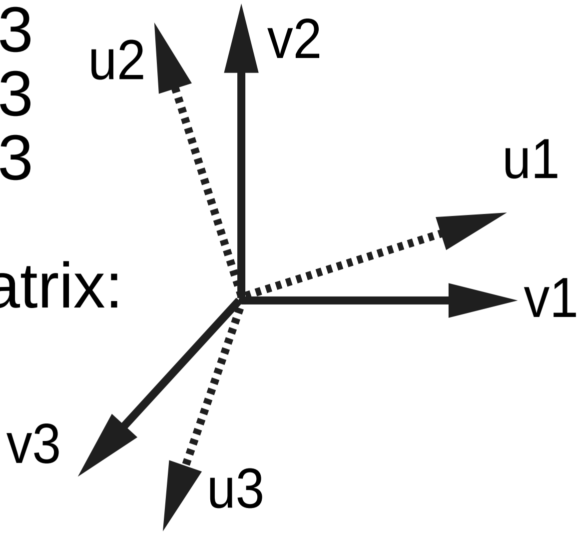
Object → ***World*** → ***View*** → ***Screen***

How the representation of a vector changes when we change the basis vectors?

$$v = (v1, v2, v3) \quad \longleftrightarrow \quad u = (u1, u2, u3)$$

There exists a relationship between the two bases:

$$u1 = \gamma 11\ v1 + \gamma 12\ v2 + \gamma 13\ v3$$
$$u2 = \gamma 21\ v1 + \gamma 22\ v2 + \gamma 23\ v3$$
$$u3 = \gamma 31\ v1 + \gamma 32\ v2 + \gamma 33\ v3$$



Which is captured by a 3x3 matrix:

$$M = \begin{bmatrix} \gamma 11 & \gamma 12 & \gamma 13 \\ \gamma 21 & \gamma 22 & \gamma 23 \\ \gamma 31 & \gamma 32 & \gamma 33 \end{bmatrix}$$

transforms a vector in one basis to its representation in the second basis:

$$u = M\, v \qquad \text{and} \qquad v = M^{-1} u$$

Moving from abstract vectors to working with column matrices of scalars

Consider a vector

$$w = \alpha 1\ v1 + \alpha 2\ v2 + \alpha 3\ v3$$

$$= a^T \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} \quad \text{where} \quad a = \begin{bmatrix} \alpha 1 \\ \alpha 2 \\ \alpha 3 \end{bmatrix}$$

while with respect to a different basis

$$u = \begin{bmatrix} u1 \\ u2 \\ u3 \end{bmatrix}$$

$$w = \beta 1\ u1 + \beta 2\ u2 + \beta 3\ u3 = b^T u$$

the 2nd basis in terms of the 1st:

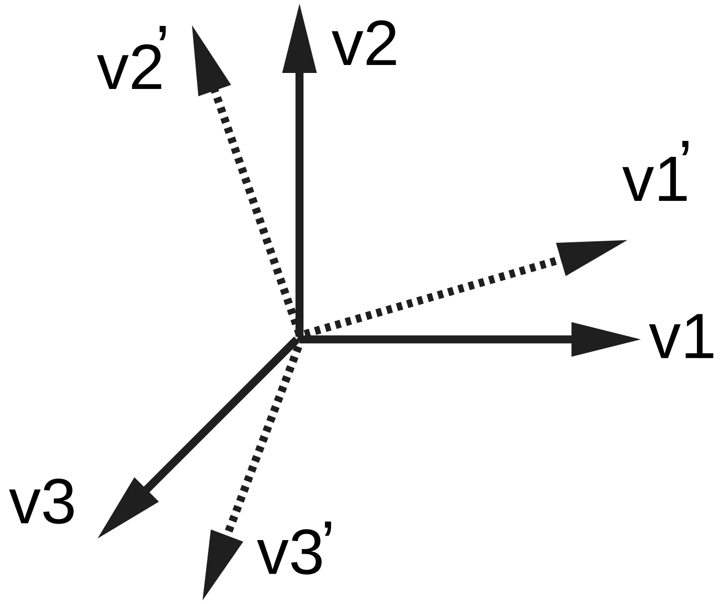
$$\begin{aligned} w &= b^T u \\ &= b^T M\, v = a^T v \end{aligned}$$

Hence

$$a = M^T b \quad \text{and} \quad b = (M^T)^{-1} a$$

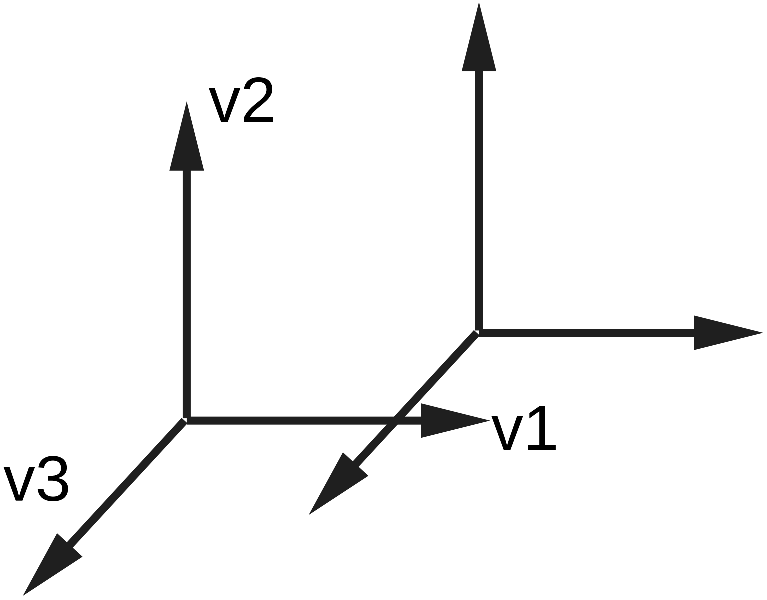
Remember the underlying basis!!

Rotation and scaling of a basis:



Leave the origin unchanged!!

Translation of a basis cannot be represented with the 3x3 matrix:



Can we "expand" the representation such that we can change frames yet still to use matrices to represent the change?

Homogeneous Coordinates

Usually we represent a point located at (*x, y, z*) using a frame determined by P_0, v_1, v_2, v_3 as

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for a vector $w = \delta_1 v_1 + \delta_2 v_2 + \delta_2 v_3$

$$w = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

How to distinguish between points and vectors?

How can we continue using matrix multiplication in 3 dimensions to represent a change in frames?

Homogeneous Coordinates!!

Use 4−d column matrices to represent both points and vectors in 3 dimensions.

Homogeneous Coordinate Representation

$$P = [\ \alpha_1\ \ \alpha_2\ \ \alpha_3\ \ 1\ ] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_2 v_3 + P_0$$

$$w = [\ \delta_1\ \ \delta_2\ \ \delta_3\ \ 0\ ] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = \delta_1 v_1 + \delta_2 v_2 + \delta_2 v_3$$

$$P = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{bmatrix} \qquad w = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ 0 \end{bmatrix}$$

1. Operating using ordinary matrix algebra
2. All affine transformation can be represented as matrix multiplication
3. Concatenation of successive transformations resulting in more efficient calculations
4. Hardware implemented

So for any two frames defined by

(v_1, v_2, v_3, P_0) and (u_1, u_2, u_3, Q_0)

A matrix representation of the change of frames:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

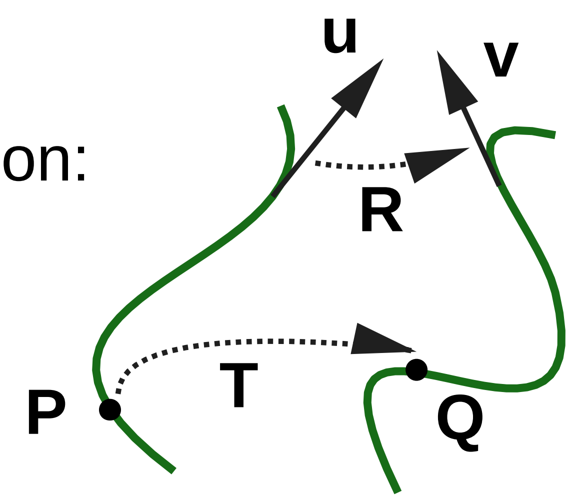
$$\text{where} \quad M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

for any two points or two vectors a and b
in the two frames in homogeneous−
coordinate representation:

$$b^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = b^T M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = a^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

Hence $\mathbf{a} = \mathbf{M}^T \mathbf{b}$

Affine Transformation



A transformation is a function:

Q = T (P)

v = R (u)

using homogeneous coordinates,
we can define transformation with
a single function:

q = f(p)

v = f(u)

Make f() a linear function:

such that $f(\alpha p + \beta q) = \alpha f(p) + \beta f(q)$

f is an affine transformation:

1. a combination of linear transformations
2. transform only end points of a line to determine completely a transformed line
3. parallel lines are transformed into parallel lines

A line in point−vector form:

$p(\alpha) = p0 + \alpha v$

For any affine transformation matrix A:

$Ap(\alpha) = Ap0 + \alpha Ad$

A line in two−point form:

$P(\alpha) = \alpha\, p0 + (1-\alpha)P1$

For any affine transformation matrix A:

$AP(\alpha) = \alpha\, Ap0 + (1-\alpha)AP1$

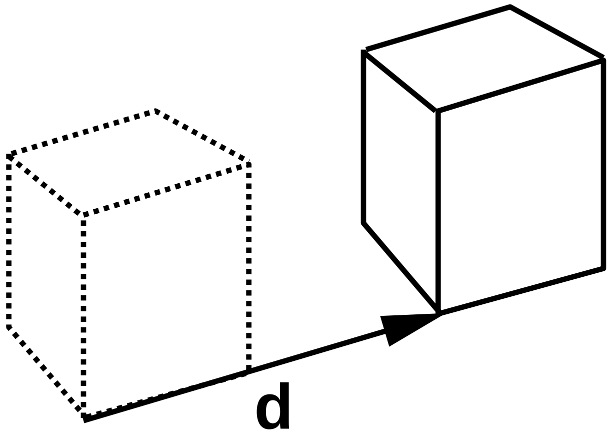
Affine Transformations preserve parallelism of lines

Rotation, translation, reflection
Scaling
Shearing

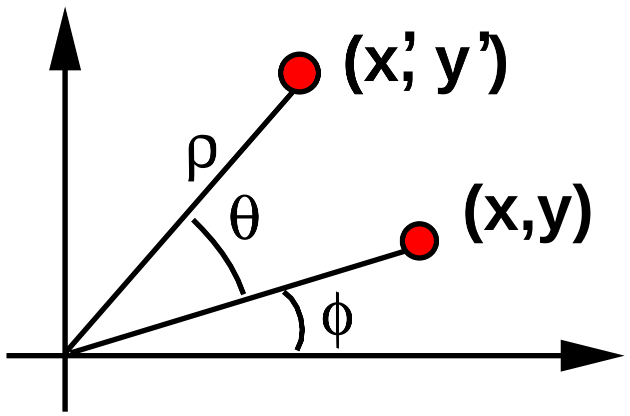
Translation

Displace points by
a fixed distance
in a given direction

$P' = P + d$



Rotation:



$x = \rho \cos\phi$
$y = \rho \sin\phi$

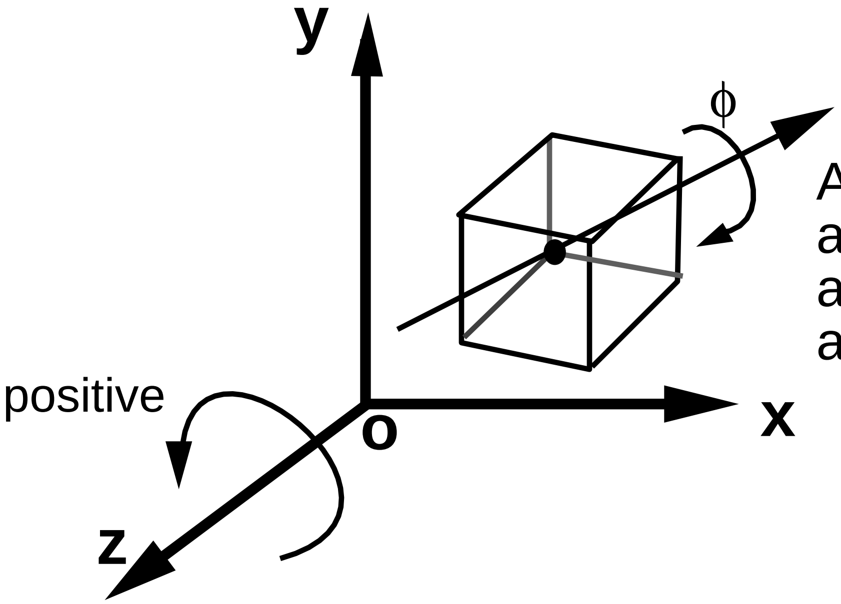
$x' = \rho \cos(\theta + \phi)$
$y' = \rho \sin(\theta + \phi)$

$x' = \rho \cos\phi \cos\phi - \rho\sin\phi \sin\theta$
$\quad = x \cos\theta - y \sin\theta$

$y' = \rho \cos\phi \sin\phi + \rho\sin\phi \cos\theta$
$\quad = x \sin\theta + y \cos\theta$

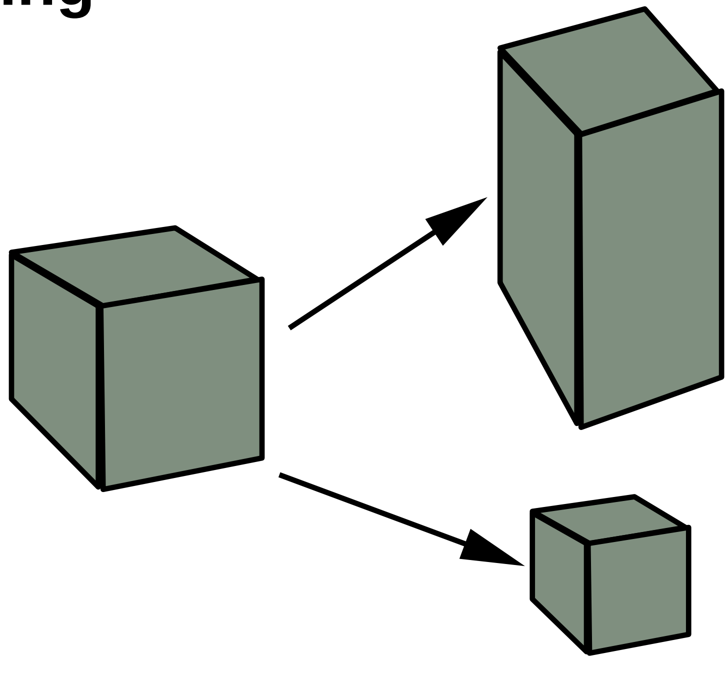
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation (cont’d)



A fixed point,
an angle, and
a line or a vector
about which to rotate

Scaling



With a fixed point, a vector, and
a scaling factor

$p' = S\,p$

Transformations in Homegeneous Coordinates

Each affine transformation is represented as a 4x4 matrix

$$M = \begin{bmatrix} \alpha 11 & \alpha 12 & \alpha 13 & \alpha 14 \\ \alpha 21 & \alpha 22 & \alpha 23 & \alpha 24 \\ \alpha 31 & \alpha 32 & \alpha 33 & \alpha 34 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A point is a column matrix

$$P = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

Translation

$$T = \begin{bmatrix} 1 & 0 & 0 & \alpha_x \\ 0 & 1 & 0 & \alpha_y \\ 0 & 0 & 1 & \alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$p' = T\,p$

$x' = x + \alpha_x$
$y' = y + \alpha_y$
$z' = z + \alpha_z$

Inverse?

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -\alpha_x \\ 0 & 1 & 0 & -\alpha_y \\ 0 & 0 & 1 & -\alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

$$S = \begin{bmatrix} \beta_x & 0 & 0 & 0 \\ 0 & \beta_y & 0 & 0 \\ 0 & 0 & \beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$p' = S\,p$

$x' = \beta_x\, x$
$y' = \beta_y\, y$
$z' = \beta_z\, z$

Inverse?

$$S^{-1} = \begin{bmatrix} 1/\beta_x & 0 & 0 & 0 \\ 0 & 1/\beta_y & 0 & 0 \\ 0 & 0 & 1/\beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What happens if $\beta_x = \beta_y = \beta_z = -1$?

Rotation about a main coordinate axis:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$p' = R_z\, p$

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z; \end{aligned}$$

Inverse?

$$\mathbf{R}^{-1}(\theta) = \mathbf{R}(-\theta) = \mathbf{R}^{T}(\theta)$$

Concatenation of Transformations

Concatenate sequences of basic transformations to define an arbitrary transformation directly

Matrix product is associative

$$\mathbf{q = CBAp}$$

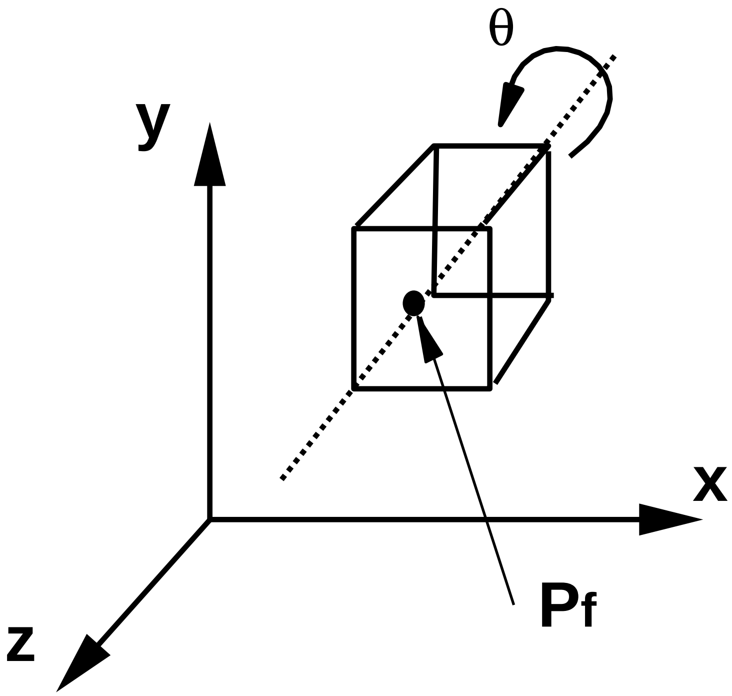
$$\mathbf{q = C(B(Ap))}$$

For transforming many points

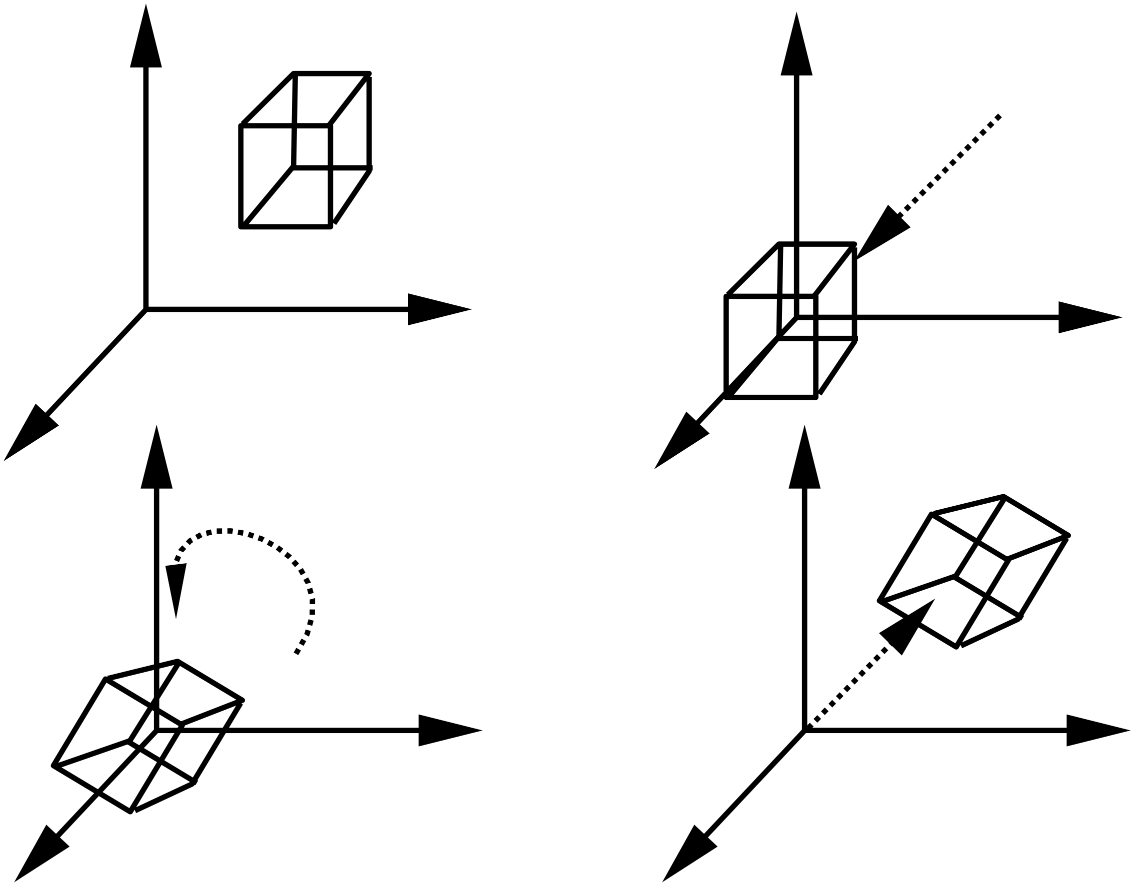
$$\mathbf{M = CBA}$$

$$\mathbf{q = M\,p}$$

Rotation About a Fixed Point



$$M = T(p_f)\, R_z(\theta)\, T(-p_f)$$



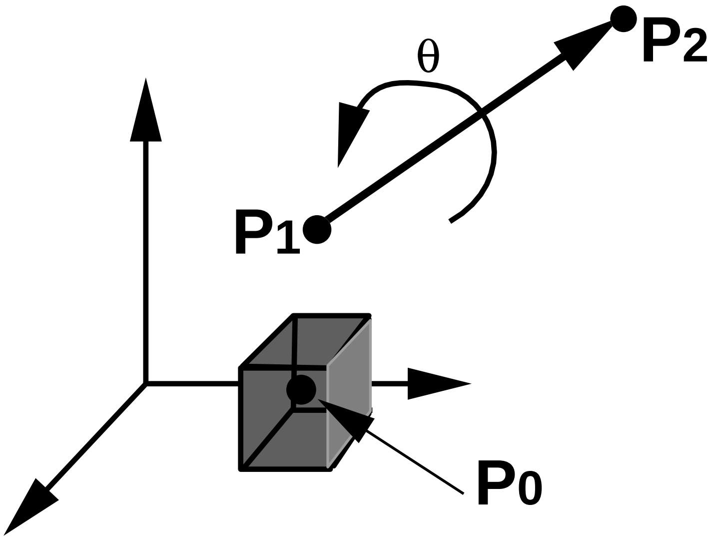
In general $$M = T(p_f)\, R(\theta)\, T(-p_f)$$

General rotation is easy
if θ_x, θ_y, θ_z are known!

Rotation About an Arbitrary Axis

To specify a rotation, we need:

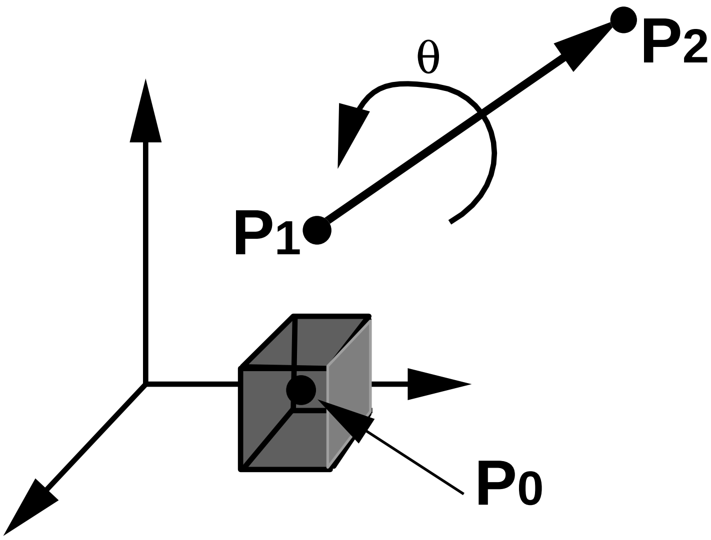
1. a fixed point P_0, center of rotation
2. a vector about which we rotate
3. an angle of rotation

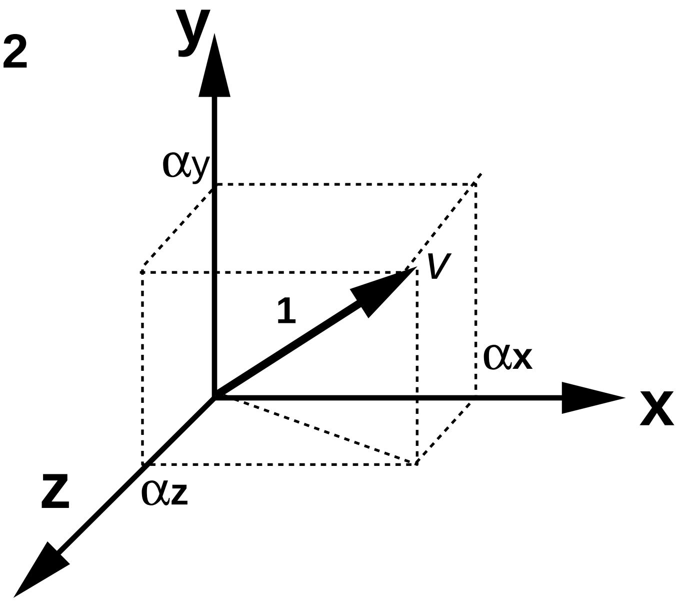


$$M = T(p_0)\, R(\theta)\, T(-p_0)$$

What is $R(\theta)$?

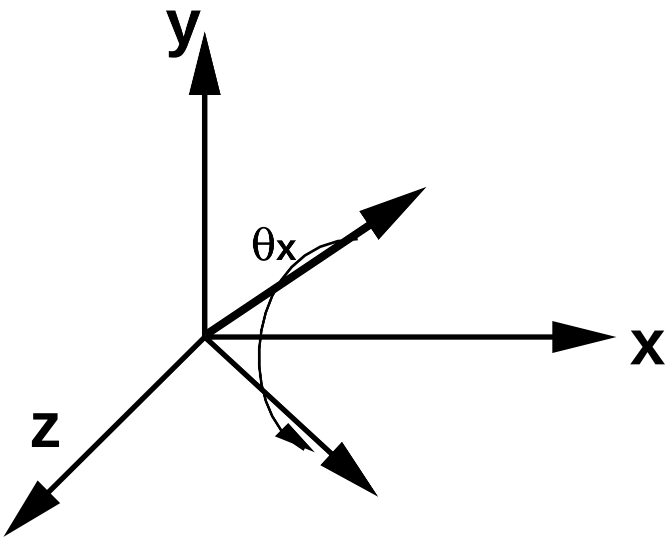
Align the axis of rotation with one of the main coordinate axis!!

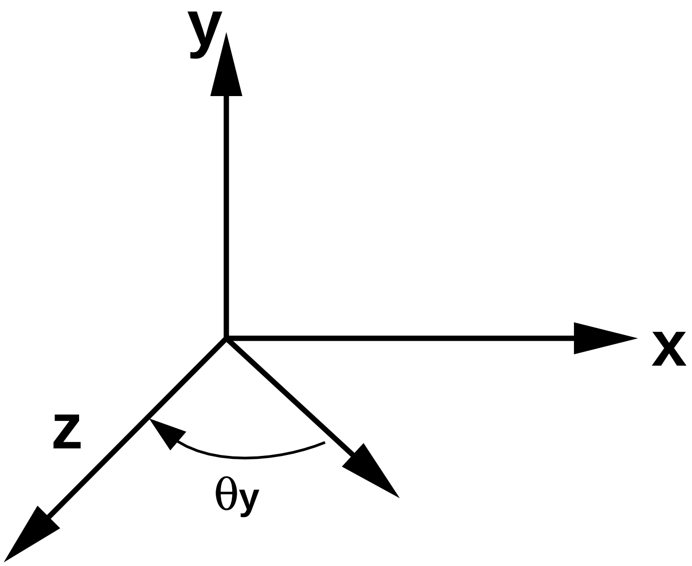




$$u = p2 - p1$$

$$v = u / |u| \qquad \alpha_x^2 + \alpha_y^2 + \alpha_z^2 = 1$$



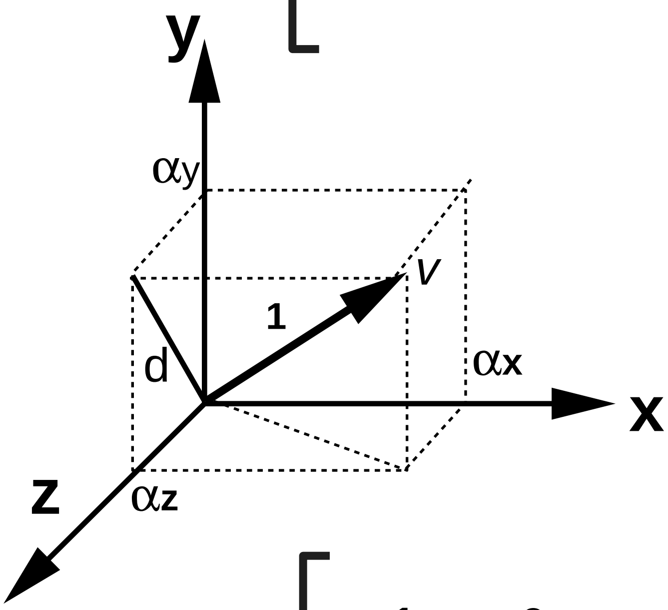


$$R(\theta) = R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x)$$

$$\boxed{M = T(p_0)\, R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x)T(-p_0)}$$

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta_y) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_z/d & -\alpha_y/d & 0 \\ 0 & \alpha_y/d & \alpha_z/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta_y) = \begin{bmatrix} d & 0 & -\alpha_x & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$