Lecture 6, 4.16.2009

Today:

Review: Basic Set Operation:

Recall the basic set operator, $\in$. From this operator come other set quantifiers and operations:

$\subseteq$, $\supseteq$

$\subsetneq$, $\supsetneq$

$\cup$, $\cap$

$\setminus$ "Set difference" (sometimes denoted – , a minus sign)

$\oplus$ - symmetric difference (xor for sets, denoted Δ in your book and sometimes denoted (circled v))

x – Cartesian product – A x B is the set of ordered pairs (a,b) with the 1st element in the first set and the 2nd element in the 2nd set

New:

P – Power set operator, unary operator (takes 1 input). ***P***(x) is the "set of all subsets of x"

P(X) = {A:A$\subseteq$X}

Lecture 6, April 16 2009

----
1. Counting, Finite/Infinite Sets (Chap 1.6)

If A is a finite set and B is a finite set, then A U B is a finite set, and further,

lA U Bl = lAl + lBl - lA \cap Bl .

Above is the **Inclusion-Exclusion principle** (we include all of A and B, then exclude those in both that are double counted).

The book extends this to triples of sets (Corollary 1.10) and more broadly it can be extended to any number (with increasingly complex combinations of parts included and excluded).

----
2. Sets of Sets, Power Sets and Partitions (1.7)

Let S = {a,b,c,d,e}

Let T contain subsets of S that contain two consecutive letters:

T = { {a,b}, {b,c}, {c,d}, {d,e}}

Note that ITI = 4.

The **Power Set** of a set S, P(S) is a set that contains all subsets of S.

For S={a,b,c},   P(S) = { {}, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c} }

note that IP(S)I = 2^ISI (each element of ISI can be in a set or not). Thus the power set is also denoted as 2^S.

We can represent any subset of a base set U = (x_1, x_2, ..., x_n} using an *index vector* of length n. The ith element of the index vector for a subset T is 1 iff x_i is in S. For example, if we use our set S above as U, then the index vector for {a,c} is 101. Note that there are 2^n possible index vectors (since each position can be 0/1), and this represents the 2^n elements of P(U).

More on power sets:

For example, take X={0,1,2}, then
$\boldsymbol{P}(x)=\{\phi,\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\}$
We can more systematically enumerate the elements of this set by counting in binary just as in a truth table, indicating if the given element is (1) or is not (0) in the given set:

"2"	"1"	"0"	
0	0	0	ϕ
0	0	1	{0}
0	1	0	{1}
0	1	1	{0,1}

1	0	1	{0,2}
1	1	0	{1,2}
1	1	1	{0,1,2}

(Aside: you should structure our truth tables by counting in binary in the usual order, like above, as good practice for readability and to be systematic)

You can also view that in the above table we have a vector of binary values where position i is 1 if the element is in the set, 0 if it is not.

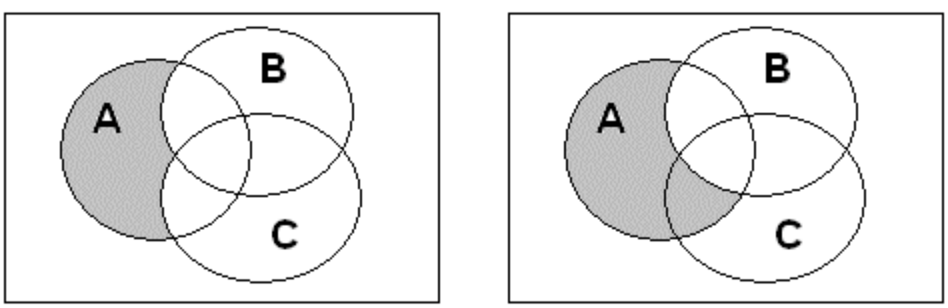
Notice in the table that the power set of X had 8 values in it. In general, if X is finite, then $|\boldsymbol{P}(X)| = 2^{|X|}$. Indeed sometimes $\boldsymbol{P}(X)$ is written 2^X as a reminder of how big this set is.

As another example, we talked about $\boldsymbol{P}(N)$, where N is the set of natural numbers.

Using the other set operators (this is mostly stuff not done in class, just extra examples):

Next, to help improve or facility with the operators on sets, we asked if the following claim is true or false:

$$(A \setminus B) \setminus C \stackrel{?}{=} A \setminus (B \setminus C)$$



We can see from the diagram that the statement is *false*. Instead, based on inspection, it would appear that:

$$(A \setminus B) \setminus C = A \setminus (B \cup C)$$

Proof: It is common to show set-equalities by arguing both inclusions. But here we will do it all at once:

$x \in (A \setminus B) \setminus C$ iff $x \in A \setminus B \wedge x \notin C$ iff
$(x \in A) \wedge (x \notin B) \wedge (x \notin C)$ iff $x \in A \wedge \neg x \in B \wedge \neg x \in C$ iff
$x \in A \wedge \neg(x \in B \vee x \in C)$ iff $x \in A \wedge \neg(x \in B \cup C)$ iff
$\mathrm{x} \in A \setminus (B \cup C)$

Set vocabulary definitions

A few more words we often use:

- singleton set – a set with only one element.
- the empty set is the set with no elements. Denoted ∅. Note this symbol is not the same as a Greek letter phi. (Indeed I believe the origin is Norwegian).
-

Partitions of Sets

For a set S, a partition of splits the elements into subsets S_1, S_2, ..., S_k such that each element of S is in exactly one of the subsets S_i (and no S_i is empty).

Alternately, S_1 U S_2 U... U S_k = S, and for any i,j S_i \cap S_j = {}

The S_i are called **cells** (sometimes).

For our set S above, if we split into vowels and consents:

P= { {a,e}, {b,c,d} } we get a partition of size 2.

Induction: 1.8

Special property that goes with N = {1,2,3, }

Mathematical Induction: Let P be a proposition define on numbers i in N: that is P(i) is True or False for each i in N (e.g. P(i) is true if i is even).

Now suppose P has the following two properties:

i) P(1) is true // this is know as the *Basis*
ii) P(k) -> P(k+1). (for k >= 1) // This is known as the I*nduction Step*

Then P(i) is true for all i in N.

Often use more general (or less structured) induction in CS.

Example in book. Another example: IP(S)I = 2^ ISI : (P(i) is for ISI=i, IP(S)I = 2^i = 2^ ISI

Basis: (i) ISI=1, so P(S) = { {}. S}, IP(S)I = 2 = 2^1 = 2^ ISI

Induction Step ii) Assume P(k), **(Induction Hypothesis)** so for ISI=k, IP(S)I = 2^k = 2^ ISI.

Now show P(k+1) (using the assumption that P(k)). Recall from our discussion of Arguments in Logic that: P(k) AND P(k) -> P(k+1) I- P(k+1)

To show P(k+1) let S = {x_1, x_2, ..., x_k+1} for P(S) we can consider two types of subsets: those that include x_k+1 and those that don't. Let A be those not using x_k+1 and B be those that use x_k+1. Claim: A = P({x_1, ..., x_k}), and thus IAI = 2^k (by our Induction Hypothesis).

Claim: each set in B is of the form T U x_k+1 (alternately, if we remove x_k+1 from any set in B, we get a set in A). thus IBI = IAI, so
IP(S)I = 2 * IAI= 2* 2^k = 2^(k+1) = 2^ ISI

Induction II (strong induction):

i) P(1) is true
ii) (For all i < k, P(i)) -> P(k).

Then P(i) is true for all i in N.

Some formal language theory

Σ - an <u>alphabet</u> – an alphabet is a finite, nonempty set. Eg, {0, 1}, {a, b}, {0,1,2,3,4,5,6,7,8,9}, {1}, ASCII – these are all alphabets. Elements in an alphabet are called *characters* (or sometimes *symbols*).

- Is ∅ an alphabet? NO, it is empty. An alphabet must be nonempty.

- is **N**, the set of natural numbers, an alphabet? No, it is infinite. An alphabet must be finite.

A string is a finite sequence of characters. The characters all come from some understood alphabet.

Examples and more definitions

Σ^* represents **all** the strings over the alphabet Σ. This is an infinite set. But all the elements in this infinite set are strings of finite length. If

$$\Sigma = \{0,1\}$$

Then

$$\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots, 111, 0000, \cdots\}$$

In this case, ε represents the "empty string". It is the unique string of length 0. In the case of the English alphabet:

$\Sigma = \{a, \ldots, z\}$

Σ^* contains dog, fish, ε , *etc*

Suppose x and y are strings, $x, y \in \Sigma^*$,

- Then $x \circ y = xy$ denotes the characters of x, in order, followed by the characters of y, again in order This is known as the concatenation operator; we have concatenated the two strings.

Consider x = dog, y = fish, then xy = dogfish.

- The symbols | … |, when applied to strings, as in $|x|$, denotes the *length* of the string x. So the symbol has a different meaning when applied to sets and strings.
- True or False? A string exists of length ∞. False, strings are finite.

- True or False? There exists a unique string of length 1. The answer depends upon the set. For $\Sigma = \{1\}$, the unary alphabet, it's true. For alphabets with 2 or more characters, it is false.

- The exponential operator in strings is defined as in $1^3 = 111$, or more generally, $x^n = x\,x^{n-1}$ for $n>0$ and $x^0 = \varepsilon$. We define the 0^{th} power as the empty string, is so the last statement holds true for n=1, i.e. $x^1 = x \circ \varepsilon = x$.