Lecture 9: 4/28/2009

Announcements: Ps5 out today: **Review Problems, not to be turned in**
Ps4 solutions out tomorrow;
Midterm Tues, May 5; Open Book and Notes; Coverage through relations (today)
Discussion this Friday: Review session (run by me). Bring questions

Relations Continued.

Reflexive: x R x for all x
Symmetric: x R y -> y R x for all x,y
Transitive: x R y and y R x -> x R z for all x, y, z

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1. Equivalence classes, quotients

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if R has all three properties, R is said to be an **equivalence** relation (2.8)

Important definition:
If R is an equivalence relation on A x A then [x] denotes the set of all elements related to x:

[x] = { x': x R x'}

We call [x] the ***equivalence class*** (or ***block***) of x.

The set of all equivalence classes of A with respect to a relation R is denoted A/R "quotient set of A by R", or "A mod R".

Claim: every equivalence relation *partitions* the universe into its blocks?

What does this mean?
Recall we defined a partitioning of the set A:

Def: { A_i: i in I} is a partition of A if each A_i is nonempty, their union is A, and they are pairwise disjoint.

Note: you can talk about the blocks being related
to one another by R, [x] R [y] iff x R y.
Well-defined.

Now go back to prior examples and identify the blocks in each case.

Proposition: Let R be an equivalence relation on a set A.
Then the blocks of R are a partition of A. (Thm 2.6 in book)

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Proof: -Every element x of A is in the claimed partition: $x \in [x]$.
-Suppose that [x] and [y] intersect. I need to argue that they are identical.

So *suppose* there exists a s.t. $a \in [x]$ and $a \in [y]$.
I must show that [x] = [y].

So suppose b in [x]. Must show $b \in [y]$.
b in [x] -> x R b is true -> b R x is true
know: x R a is true -> b R a is true -> a R b is true
know a R y is true -> y R a is true -> y R b is true -> b in [y]

Suppose b in [y]. Must show b in [x].
Exactly analogous.

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Proposition: Suppose you have a partition $\{ A_i \ i \in I\}$ of A.
Then this determines a equivalence relation in the natural way; x R y iff $x \in A_i$ and $y \in A_i$ for some i.

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Go back to prior examples and identify the blocks:

Example 1. alpha E beta if alpha, beta regular expressions denoting the same language

--> each block is the set of regular expressions denoting the same language .
Can you think of a "canonical" name for each block?

Could use the lexicographically first regular expression in a block.
Sounds hard to find the canonical name for a block (that is, given alpha, find the name of [alpha].

Example 2. strings x and y are equivalent if they have the same length. YES

--> blocks are the strings of a given length. Canonical name: 0^i.

Example 3. Fix a DFA (like the one we looked at for even a's and even b's). x~y if they go to the same state under M. YES

--> blocks: do a small example, like the DFA that checks if inputs and outputs have the same length.

(NOT done in class) Example 4: given a regular language L, say x E y if (xz in L iff yz in L for all z)
What are the blocks??

Example 5. words x and y are equivalent if they are anagrams of each other (R is a relation on words in a dictionary). Example: trap, tarp, part, rapt in the same equivalence class.

--> blocks are the words that anagram to each other. Canonical name: alphabetically first word (part) or could be letters in alphabetical order (aprt) NOTE the later is not in the set of words.

Example 6: For integers a,b, a $=_m$ b IFF (a mod m) = (b mod m) IFF m | a-b

NOT DONE In Class: Example 7: Let U be the set of people, x B y have the same birthday.
What are the blocks of U/B ?
How many elements are there in U/B (366)

NOT DONE In Class Example 8: Let \R be the set of real numbers, x~y if \lfloor x \rfloor =
\lfloor y \rfloor
What are the blocks? What would be a "canonical element" (name) for each block?

Example 9: \Let \R be the set of real number, x~y if
x - \lfloor x \rfloor =
y - \lfloor y \rfloor
What are the blocks? What would be a "canonical element" (name) for each block?
Real values in the range (-1, 1).

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2. Closures of Relations (2.7)

Given a relation, we sometimes want to FORCE it to be reflexive ... or symmetric ... or transitive ...or any combination of these.

We do this by adding _just_ enough extra pairs to get the property.

Draw a picture for the transitive closure of a directed graph.

Draw a picture for reflexive-transitive closure.

Def: The transitive closure of a relation R on A x A
is the smallest relation R* that contains A and is transitive.

claim: this is well defined: there is a unique smallest relation R   that contains A x A and is transitive.
In fact, it is the intersection of all relations R that are transitive and contain A.

Procedure to find the transitive closure:
"add in all shortcuts". In the graph-theoretic realization
of the relation R, just add an arc (a,c) whenever you
have an (a,b) and a (b,c) but lack an (a,c).

Example 1: Do it with a graph

Example 2: An infinite set example:
Suppose i R i+1 for all i in Z.
Transitive closure is what?

Another view of Transitive closure is to consider $R^2 = R \circ R$, which contains all pairs implied directly by transitivity, then $R^3 = R^2 \circ R$, which contains all pairs implied by transitivity used twice, and $= R^4$ and so on, If we consider all powers up to $= R^{n-1}$, and union them all together, we get the transitive closure of a finite set of size n.

NOTE DONE in Class: Let |- be a relation on states of your computer, where the state
captures all the contents of your machine: registers, disk contents,
etc. Assume not hooked up to the web or a keyboard so that
whatever happens next is determined by its present state

|+
|--- for the transitive closure of the above. What would this represent?
| You will go from state C to state C'. Eventually.

What would you say if C |- C? Your machine is looping.

Reflexive closure R <- R union {(x,x): x in A} (easy)

Transitive closure: if (x,y) in R and (y,z) in R add (x,z) to R.