Problem Set 1 – Due Thursday, October 4, 2012

Instructions: Write up your solutions as clearly and succinctly as you can. Typeset solutions, particularly in LaTeX, are always appreciated. Don't forget to acknowledge anyone with whom you discussed problems. Recall that homeworks are due at 10:10 am on Thursdays, in the turn-in box in Kemper Hall, room #2131.

Problem 1. For any $n \geq 0$, consider the graph $H_n = (V_n, E_n)$ whose vertices are $\{\texttt{0}, \texttt{1}\}^n$ and with an edge $\{x, y\} \in E_n$ iff x and y differ at a single bit position.

(a) Find an alternative, *recursive* definition for these same graphs.

(b) Draw pretty pictures of H_0, H_1, H_2, H_3, H_4. Your picture should mirror the recursive formulation of these graphs.

(c) Establish formulas for $v(n) = |V_n|$ and $e(n) = |E_n|$. (The second will require proof.)

(d) If L and L' are lists of strings and x is a string, let L^{flip} be the list L in reverse order, let $L + L'$ be all the list of elements in L followed by the list of elements in L', and let xL be the list of elements in L with each preceded by the string x. Consider the sequence of lists $L_0 = (\varepsilon)$ and $L_n = 0L_{n-1} + 1L_{n-1}^{\text{flip}}$ for $n \geq 1$. Enumerate the elements of the lists L_0, L_1, L_2, L_3, L_4. Do you find anything "interesting" about the list L_4, say?

(e) Recall that a graph is *Hamiltonian* if there is a cycle that visits each of its vertices once and only once. Prove that H_n is Hamiltonian for all $n \geq 2$.

Problem 2. For each of the following counting problems, give not only your answer, but explain how it is computed, justifying any formulas employed. None of these should be overly tedious or require computer calculation.

(a) Let $L = \{\texttt{a}, \texttt{b}, \texttt{ab}\}$. How many strings of length 10 are in L^*?

(b) Let $L = \{\texttt{a}, \texttt{bb}\}$. How many strings of length 10 are in L^*?

(c) Let $L = \{\texttt{a}, \texttt{ab}, \texttt{ba}\}$. How many strings of length 10 are in L^*?

Problem 3. State whether the following propositions are true or false, carefully explaining each answer.

(a) $\emptyset^*$ is a language.

(b) ε is a language.

(c) Every language is infinite or has an infinite complement.

(d) Some language is infinite and has an infinite complement.

(e) The set of real numbers is a language.

(f) There is a language that is a subset of every language.

(g) The Kleene closure (the star) of a language is always infinite.

(h) The concatenation of an infinite language and a finite language is always infinite.

(i) There is an infinite language L containing the emptystring and such that L^i is a proper subset of L^* for all $i \geq 0$.

Problem 4. Give DFAs for the following languages. Assume an alphabet that includes all and only the mentioned symbols. Make your DFA as small as possible.

(a) The set of all strings that have `abba` as a substring.

(b) The set of all strings that do not have `abba` as a substring.

(c) The complement of $\{0, 01\}^*$.

(d) The set of all strings that have an even number of 0's and an even number of 1's.

(e) The binary encodings of numbers divisible by 5. Allow leading zeros.

Problem 5. Fix a DFA $M = (Q, \Sigma, \delta, q_0, F)$. For any two states $q, q' \in Q$, let us say that q and q' are *equivalent*, written $q \sim q'$, if, for all $w \in \Sigma^*$ we have that $\delta^*(q, w) \in F \Leftrightarrow \delta^*(q', w) \in F$. Here δ^* is the extension of δ to Σ^* defined by $\delta^*(q, \varepsilon) = q$ and $\delta^*(q, ax) = \delta^*(\delta(q, a), x)$.

(a) Prove that $\sim$ is an equivalence relation.

(b) Suppose that $q \sim q'$ for distinct q, q'. Describe, first in plain English and then in precise mathematical terms, how to construct a smaller (=fewer state) DFA M' that accepts the same language as M.