Problem Set 9 – Due Friday, May 29, 2015

Problem 1. As you did last week, classify each of the following languages as **recursive**, **r.e.** but not decidable, **co-r.e.** but not decidable, or **neither** r.e. nor co-r.e. Giving reductions where appropriate, prove your results. *This problem is very important. Understanding reductions may be the most important concept of ECS 120.*

1.1 $A = \{\langle M, k\rangle : M$ is a TM that accepts at least one string of length $k\}$.

1.2 $B = \{\langle M, k\rangle : M$ is a TM that runs forever on at least one string of length $k\}$.

1.3 $C = \{\langle M, k\rangle :\ M$ is a TM that accepts a string of length k and diverges on a string of length $k\}$. Assume that the underlying alphabet has at least two characters.

1.4 $D = \{\langle M\rangle :\ M$ is a TM that accepts some palindrome$\}$.

1.5 $E = \{\langle G_1, G_2\rangle :\ G_1$ and G_2 are CFGs and $L(G_1) \oplus L(G_2) = \emptyset\}$.
You may assume that $L = \{\langle G\rangle :\ G$ is a CFG and $L(G) = \Sigma^*\}$ is undecidable.

1.6 $F = \{\langle M\rangle :\ M$ is a TM and $L(M)$ is recursive$\}$.

Problem 2 Prove or disprove each of the following claims.

2.1 $A \leq_{\mathrm{m}} A$.

2.2 If $A \leq_{\mathrm{m}} B$ and $B \leq_{\mathrm{m}} C$, then $A \leq_{\mathrm{m}} C$.

2.3 If $A \leq_{\mathrm{m}} B$ then $\overline{A} \leq_{\mathrm{m}} \overline{B}$.

2.4 If A is recursive, then $A \leq_{\mathrm{m}} a^*b^*$.

2.5 If $A \leq_{\mathrm{m}} B$ then $B \leq_{\mathrm{m}} A$.

2.6 If $A \leq_{\mathrm{m}} B$ and $B \leq_{\mathrm{m}} A$ then $A = B$.

Problem 3. Let us say that a nonempty set B is *countable* if you can list (possibly with repetitions) its elements $B = \{a_1, a_2, a_3, \ldots\}$; more formally, there is a surjective[1] function f from $\mathbb{N}$ to B. We'll say that the empty set is also countable. A set is *uncountable* if it is not countable.

3.1 Prove that any subset A of a countable set B is countable.

3.2 Fix an alphabet Σ. Prove that there are countably many finite languages over Σ.

3.3 Fix an alphabet Σ. Prove that there are uncountably many infinite languages over Σ.

[1] Recall that a function $f : A \to B$ is surjective (or onto) if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$.