Quiz 9 Solutions

For this quiz I want you to prove that

$$A = \{\langle M, k\rangle : M \textit{ is a TM that accepts at least one string of length } k\}$$

is undecidable. Do this with a reduction involving A_{TM} *or* $\overline{A_{\text{TM}}}$. *Make your proof succinct, legible, and logical. Write exclusively in grammatical English sentences.*

Setup. Since A is r.e., we will show that it is undecidable by showing that $A_{\text{TM}} \leq_{\text{m}} A$. To do this, we must construct a Turing-computable function that maps a string $\langle M, w\rangle$ to a string $\langle M', k\rangle$ such that TM M accepts w if and only if TM M' accepts some string of length k.

Construction. Given $\langle M, w\rangle$ the reduction returns $\langle M', k\rangle$ where $k \geq 0$ is an arbitrary fixed value, say $k = 0$, and TM M' is the following machine:

Machine M', on input x:
  Run M on w
  If M accepts then accept
  If M rejects then reject

Analysis. If M accepts w then we will have that $L(M') = \Sigma^*$, so M' will accept a string of length k (as it accepts all strings of all lengths). On the other hand, if M does not accept w then $L(M') = \emptyset$ so M' will not accept any string of length k (as it accepts no string of any length). Finally, the function that computes $\langle M', k\rangle$ from $\langle M, w\rangle$ is clearly computable.