Problem Set 1 – Due Tuesday, January 17, 2012

Instructions: Write up your solutions as clearly and succinctly as you can. Typeset solutions, particularly in LaTeX, are always appreciated. Don't forget to acknowledge anyone with whom you discussed problems. Recall that homeworks are due at 1:00 pm sharp on Tuesdays, in the turn-in box in Kemper Hall, room #2131.

Problem 1. Show that at a party of 20 people, there are at least two people who have the same number of friends present. Assume, however unrealistically, that friendship is symmetric and anti-reflexive.

Problem 2. Fix a DFA $M = (Q, \Sigma, \delta, q_0, F)$. For any two states $q, q' \in Q$, let us say that q and q' are *equivalent*, written $q \sim q'$, if, for all $w \in \Sigma^*$ we have that $\delta^*(q, w) \in F \Leftrightarrow \delta^*(q', w) \in F$. Here δ^* is the extension of δ to Σ^* defined by $\delta^*(q, \varepsilon) = q$ and $\delta^*(q, ax) = \delta^*(\delta(q, a), x)$.

(a) Prove that $\sim$ is an equivalence relation.

(b) Suppose that $q \sim q'$ for distinct q, q'. Describe, first in plain English and then in precise mathematical terms, how to construct a smaller (=fewer state) DFA M' that accepts the same language as M.

Problem 3. State whether the following propositions are true or false, carefully explaining each answer.

(a) Every language is infinite or has an infinite complement.

(b) Some language is infinite and has an infinite complement.

(c) The set of real numbers is a language.

(d) There is a language that is a subset of every language.

(e) The Kleene closure (the star) of a language is always infinite.

(f) The concatenation of an infinite language and a finite language is always infinite.

(g) There is an infinite language L containing the emptystring and such that L^i is a proper subset of L^* for all $i \geq 0$.

Problem 4. Give DFAs for the following languages. Assume an alphabet that includes all and only the mentioned symbols. Make your DFA as small as possible.

(a) The set of all strings that have `abba` as a substring.

(b) The set of all strings that do not have `abba` as a substring.

(c) The complement of $\{0, 01\}^*$.

(d) The set of all strings that have an even number of 0's and an even number of 1's.

(e) The binary encodings of numbers divisible by 5. Allow leading zeros.

Problem 5 State whether the following propositions are true or false, proving each answer.

(a) Every DFA-acceptable language can be accepted by a DFA with an even number of states.

(b) Every DFA-acceptable language can be accepted by a DFA whose start state is never visited twice.

(c) Every DFA-acceptable language can be accepted by a DFA no state of which is ever visited more than once.

(d) The language $L = \{x \in \{a, b\}^* : x \text{ starts and ends with the same character}\}$ can be accepted by a DFA $M = (Q, \Sigma, \delta, q_0, F)$ for which $\delta^*(q_0, w) = q_0$ for some $w \neq \varepsilon$. Assume and alphabet of $\Sigma = \{a, b\}$.