ECS 20 — Lecture 16 — Fall 2013 —19 Nov 2013

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Today:

- Quiz 3
- Finish pigeonhole applications
- Graphs

Remaining topics: graphs, counting, and probability

Pigeonhole Principle

Recall:

Restated: [Pigeonhole principle]
If $f: A \rightarrow B$ where A and B are finite sets, $|A|>|B|$, then f is NOT injective.

Stronger form

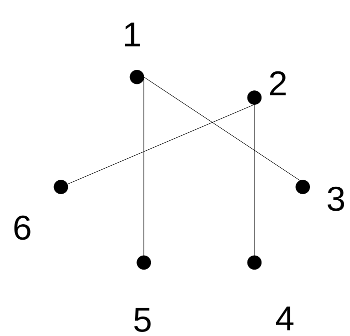
[Pigeonhole principle, strong form]
If $f: A \rightarrow B$ where A and B are finite sets, then so point $b \in B$ must have at least $\lceil |B|/|A| \rceil$ preimages.

Eg, if 100 pigeons roost in 30 holes, some hole has at least 4 pigeons roosting therein.

Ex: Given five points inside the square whose side is of length 2, prove that two are within \sqrt{2} of each other.

Soln: divide square into four 1 x 1 cells. Diameter of each cell = \sqrt{2}

Ex. (beautiful example) In any room of 6 people, there are 3 mutual friends or 3 mutual strangers (Ramsey theorem, and R(3,3)=6)



Remove person 1 5 people left.
Put into two pots: friends with 1, non-friends with 1.
One has at least three people.
If three friends: *Case 1: some two know each other: DONE*
Case 2: no two know each other: DONE
If three non-friends: ...o

Difficult Puzzle: What is the minimum number of people that must assemble in a room such that there will be at least n friends or n non-friends: R(n,n)

R(4,4) = 18 (1955)
R(5,5) = ?? open!!! known to be between 43 (1989) and 49 (1995)
R(10,10) =?? open and not tightly determined at all: range 798 (1986)- 23,556 (2002)

Cliques and Ramsey numbers

The following definitions assume graph-theory terminology, to be introduced shortly.
A clique of size n in a graph is a set of n mutually connected vertices. Denote K_n.

An independent set of size m in a graph is a set of m vertices that are mutually non-adjacent.

Clique number $\kappa(G)$ = size of a *largest* clique within G (largest subgraph that's a clique)
FACT: There is no efficient algorithm known to calculate $k(G)$.

Let $R(n,m)$ = the minimum number of vertices such that a graph of $R(n,m)$ vertices either has a clique of size n or an independent set of size m.

Alternative version:
$R(n,m)$ = the minimum number of **vertices** such that if you red/blue color the **edges** of a graph with of $R(n,m)$ vertices, there is either a **red clique** of size n or a **blue clique** of size m.

Theorem: The above is well-defined:
for every $n,m \geq 1$ there exists a smallest number $R(n,m)$ such that every graph on $R(n,m)$ vertices contains either a clique of size n or an independent set of size m.

Claim: $R(3,3)=6$

Interpretation: In any room of 6 people, there are 3 mutual friends or 3 mutual strangers.

$R(n,m)$: First, $R(3,3) \geq 6$. Draw a five pointed start surrounded by a circle.
There is no triangle and no independent set of size 3.

Second, $R(3,3) \leq 6$

Remove person 1 5 people left.
Put into two pots: friends with 1, non-friends with 1.
One has at least three people.

If three friends: Case 1: some two know each other: DONE
Case 2: no two know each other: DONE
If three non-friends: ...o

Difficult Puzzle: What is the minimum number of people that must assemble in a room such that there will be at least *n* friends or *n* non-friends: $R(n,n)$

R(4,4) = 18 (Greenwood and Gleason, 1955)
R(5,5) in [43, 49] Exoo, 1989; Radziszowski 1995
R(6,6) in [102,165]

"Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers [not computer scientists?!] and all our mathematicians and attempt to find the value.

"But suppose, instead, that they ask for $R(6, 6)$. In that case, he believes, we should attempt to destroy the aliens." Quoted by Joel Spencer

Graph theory

Graph theory
1. Basic definitions
2. Representation of graphs
3. Isomorphic graphs
4. Paths and cycles
5. Trees
6. Eulerian and Hamiltonian graphs
7. Longest and shortest paths
8. Colorability
9. Planarity
10. Cliques and Ramsey numbers

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1. Basic Definitions

Def: A (finite, simple) graph $G=(V, E)$ is an ordered pair
- V is a nonempty finite set (the *vertices* or *nodes*)
- E is a set of two-elements subsets of V (the *edges*)

There are many other "kinds" of graphs—for example, in a *directed graph* (*digraph*), the edges (now often called *arcs*) are **ordered** pairs, instead of unordered pairs. But, for this lecture, let's stick to *simple* graph, the kind that I defined.

Conventional representation: picture.
Be clear: the picture is **NOT** the graph, it is a representation of the graph.

```
    B--- C
   / \   /
  A    D

    B--- D
   / \  /
  A    C
```

are the SAME graph. Not a graph: draw a self-loop, multiple loops, empty set.

Def: Two vertices v, w of a graph $G=(V, E)$ are **adjacent** if $\{v,w\}\in E$.
Def: The **degree** of a vertex $\deg(v) = |\{v,w\}: w\in V|$

Question: what is the maximal and minimal degrees of an n-vertex graph?

I like $\{x,y\}$ for an edge, emphasizing that {x,y} are unordered.
Will sometimes see xy or (x,y), but both look like the order matters, which, in a simple graph, it does not.

Usually we use $n=|V|$ and $m=|E|$; alternatively, $\nu=|V|$ and $\varepsilon=|E|$

Prop: $\Sigma_v \deg(v) = 2m$

We don't usually care about the *names* of points in V, only how they're connected up.
That's because the properties of the graphs that matter are those that are invariant under *isomorphism*.
Two graphs $G=(V, E)$ and $G'=(V', E')$ are isomorphic if there is a permutation $\pi: V \rightarrow V'$ such that $\{v,w\}\in E$ iff $\{\pi(v),\pi(w)\}\in E'$. Explain.

Give examples of properties that are (and are not) preserved under isomorphism

How many different graphs are there on $V=\{1,...,n\}$? 2^{n choose 2} = 2^{n(n-1)/2}

2. Representation of graphs

Two common ways: *adjacency matrix* and *adjacency list*.
Describe them. Likely this was covered in 30 in or 40, which many students have had, although certainly not everyone.

3. Isomorphic graphs

Def: Graphs $G=(V,E)$ and $G'=(V',E')$ are **isomorphic** if there exists a permutation π such that $\{x,y\}\in E$ iff $\{\pi(x),\pi(y)\} \in E'$.

Amazing fact: there is no efficient algorithm known to decide if two graphs are isomorphic. (Most computer scientists believe that no such algorithm exists.) One of the biggest open questions in computer science.