ECS 20 — Lecture 17 — Fall 2013 —21 Nov 2013

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Today:

- Graph Theory, continued

Graph theory

1. Review of definitions and vocabulary
2. Sum-of-degrees formula. Cliques. Counting. Bipartite graphs.
3. Paths, cycles, Eulerian graphs

1. Basic Definitions

Def: A (finite, simple) **graph** $G=(V, E)$ is an ordered pair
- V is a finite nonempty set (the *vertices* or *nodes*)
- E is a set of two-elements subsets of V (the *edges*)

I like $\{x,y\}$ for an edge, emphasizing that $\{x,y\}$ are unordered.
Will sometimes see xy or (x,y), but both look like the order matters, which, in a simple graph, it does not.

Usually people use $n=|V|$ and $m=|E|$; alternatively, $\nu = |V|$ and $\varepsilon = |E|$ looks nice and suggestive.

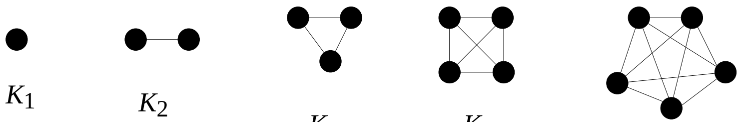
There are many other "kinds" of graphs—for example, in a **directed graph** (**digraph**), the edges (now often called **arcs**) are **ordered** pairs, instead of unordered pairs. We sometimes allow graphs with **self-loops** (and edge between a vertex and itself) or **multiple edges** (two or more different edges connecting a pair of nodes). In a network, each edge (or arcs) has a real-valued **weight**. People consider **infinite graphs**. We even have graphs where an even can be *incident* (touch) touch than two vertices (**hypergraphs**). None of these variants are not allowed in simple graphs. For this lecture, we're going to stick to them.

Conventional representation: a **picture**. (Draw some.) But be clear: the picture is **NOT** the graph, it is a **representation** of the graph. The graph is the pair (V,E).

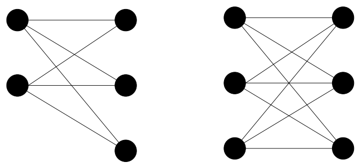
Some "special" graphs – a **clique** of size n, K_n , and complete **bipartite graphs** on n "boys" and m "girls", $K_{n,m}$.

Def: Two vertices v, w of a graph $G=(V, E)$ are **adjacent** if $\{v,w\}\in E$.
Def: The **degree** of a vertex $\deg(v) = |\{v,w\}: w\in V|$
Def: The neighbor set of v in a graph $G=(V,E)$ is $N(v) = \{w\in V: \{v,w\} \in E\}$.

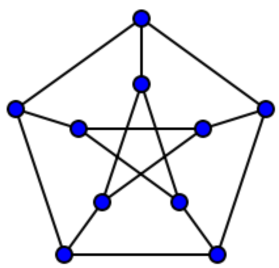
Note that deg(v) = $|N(v)|$.



K_1 K_2 K_3 K_4 K_5



$K_{2,3}$ $K_{3,3}$



The Peterson graph

Some counting
Question: How many different graphs are there on $V=\{1,\ldots,n\}$? $2^{C(n,2)} = 2^{n(n-1)/2}$
Question: what is the maximal and minimal degrees of an n-vertex graph?
Question: Count how many edges in K_n and $K_{n,m}$.

Isomorphism

We don't usually care about the *names* of points in V, only how they're connected up. If two graphs are the same, up to renaming, we call them isomorphic. Formally, graphs $G=(V, E)$ and $G'=(V', E')$ are **isomorphic** if there is a permutation $\pi: V \to V'$ such that $\{v,w\} \in E$ iff $\{\pi(v),\pi(w)\} \in E'$. The properties of graphs that matter are those that are invariant under isomorphism.

Def: Graphs $G=(V,E)$ and $G'=(V',E')$ are **isomorphic** if there exists a permutation π such that $\{x,y\} \in E$ iff $\{\pi(x),\pi(y)\} \in E'$.

Proposition: Isomorphism is an equivalence relation.

Amazing fact: there is no efficient algorithm known to decide if two graphs are isomorphic. (Most computer scientists believe that no such algorithm exists.) One of the biggest open questions in computer science.

Maybe show how to prove to graphs are *non*-isomorphic using an **interactive proof**. [Didn't do this]

Prop: $\Sigma_v \deg(v) = 2m$

Bipartite graphs

Def: A graph $G = (V,E)$ is ***k*-colorable** if we can paint the vertices using "colors" $\{1,...,k\}$ such that no adjacent vertices have the same color. Formally,

Def: A graph is **bipartite** if it is 2-colorable. In other words, we can partition V into (V_1, V_2) such that all edges go between a vertex in V_1 and a vertex in V_2.

Proposition: A graph is bipartite iff it is 2-colorable)iff it has no odd-length cycles, which can be done after we introduct that vocabulary.)

Proposition: There is a simple and efficient algorithm to decide if a graph G is 2-colorable / bipartite.

Proof: Modify DFS.

Initially, all vertices are uncolored: color[v]=UNCOLORED
While there are uncolored vertices v in G do DFS(v,0)

Algorithm DFS(v,b)
color[v] = b
for each uncolored w in $N(v)$ do DFS(w, 1-b)

Amazing fact: There is no reasonable algorithm known to decide if a graph is 3-colorable. (Most computer scientists believe that no such algorithm exists.)

Proposition [Appel, Haken 1989] Every planar graph is 4-colorable.

3. Paths, Cycles, connectivity, and Eulerian cycles

Def: A ***path*** $p=(v_1, ..., v_n)$ in $G = (V,E)$ is a sequence of vertices s.t. $\{v_i,v_{i+1}\}\in E$ for all i in $\{1,..., n-1\}$.

A path is said to *contain* the vertices and to *contain* the edges $\{v_i,v_{i+1}\}$.
The *length* of a path is the number of **edges** on it.

A ***cycle*** is a path of length three or more that starts and ends at the same vertex and includes no repeated edges.

A graph is **acyclic** if it contains no cycle.

A graph $G = (V,E)$ is **connected** if, for all x,y in V, there is a path from x to y.

The **components** of a graph are the maximal connected *subgraphs*.
(Graph $G' = (V',E')$ is a *subgraph* of graph $G = (V,E)$ if $V' \subseteq V$ and $E' \subseteq E$.)

Alternative definition of components: Say that $x \sim y$ (these vertices are in the same component) if there is a path from x to y. **Prop**: this is an equivalence relation. Its blocks (equivalence classes) are the components.

Alternative definition of a component: the component containing v is all vertices connected to v by paths of any lengths; and all the induced edges (the edges of the original graph that span vertices in the component).

Describe an algorithm, based on DFS, for counting the number of components of a graph and identifying them.

Def: A graph G is **Eulerian** if it there is a cycle C in G that goes through every **edge** exactly once.

A graph G is **Hamiltonian** if there is a cycle that goes through every **vertex** exactly once.

Theorem: (Euler) A connected graph $G = (V,E)$ on $n \geq 3$ vertices is Eulerian

iff

every vertex of G is of even degree.

Proof: ➔ Choose s. Graph is Eulerian mean there is a path that starts at s and eventually ends at s, hitting every edge. Put a label of 0 on every vertex. Now, follow the path. Every time we exit a vertex, increment the label. Every time we enter a vertex, increment the label. At end of traversing the graph, label(v) = deg(v) and this is even.

🡐 (sketch) If every vertex is of even degree, at least three vertices. Start at s and grow a cycle C of unexplored edges until you wind up back at s. You never "get stuck" by even-degree constraint. If every edge explored: Done. Otherwise, find contact point of C and an unexplored edge (exists by connectedness) and grow out from there. Splice together the paths.

So there is a trivial algorithm to decide if G is Eulerian: just check if all its vertices are of even degree.

Amazing fact: There is no "reasonable" algorithm known to decide if a graph is Hamiltonian. (Most computer scientists believe that no such algorithm exists.)