

## Problem Set 8 — Due Wednesday, November 17, at 5pm

For the following problems, do all calculations by hand—no computers or calculators. Done well, nothing here is or tedious to compute. Make sure to show all work and provide good explanations.

1. Prove that  $2^n > 10n^2$  for all sufficiently large integers  $n$ .
2. How many (positive) divisors  $d(n)$  does the number  $n = 2\,450\,250\,000$  have?
3. (a) Are 2021 and 2022 relatively prime (they have no common divisors but  $\pm 1$ )?  
(b) Compute  $\gcd(2021, 6020)$ .  
(c) How many points are in  $\mathbb{Z}_{2021}^*$ , where  $\mathbb{Z}_n^* = \{i \in [1..n] : \gcd(i, n) = 1\}$ .
4. A thousand Christmas lights, indexed  $[1..1000]$ , decorate the yard of Soraya's home. All the lights are all initially OFF. A mischievous raccoon goes to each light  $n \in [1..1000]$  and toggles the switch. Now the 1000 lights are ON. It looks pretty. Yet thinking the better of it, the raccoon decides to go back and toggle the switch for each even-indexed light. Now there will be 500 lights ON. Next our friend toggles each light whose index is a multiple of 3. After that, it toggles each light whose index is a multiple of 4. And so on. On round 1000, it toggles the switch only for light number 1000. After that, the exhausted raccoon goes to sleep. How many lights illuminate the sleeping creature? Fully explain your answer.