1 True / False [40 points: 5 points each]

Put an **X** through the **correct** box. Grading: *+5 for a correct answer; -5 for an incorrect answer; 0 for no answer; negative totals replaced by zero.*

ϕ^*

$\{\varepsilon\}^*$

A. If a language is regular then it is finite. True [redacted]

B. If A is a nonempty language then A^* is infinite. True [redacted]

C. A graph can have more components than it has edges. [redacted] False

D. A graph G has 10 vertices. The degrees of these vertices could be: 2,2,4,4,5,6,6,6,6,8. True [redacted]

E. Define the relation $\sim \subseteq \mathsf{R} \times \mathsf{R}$ by $a \sim b$ iff $ab \in \mathsf{Q}$. Then $\sim$ is an equivalence relation. True [redacted]

$\mathbb{Q}$

$\mathbb{R} \times \mathbb{R}$ \mathbb

F. $\sum_{i=1}^{n} i = \binom{n+1}{2}$. [redacted] False

$1+2+\cdots+n = \frac{n(n+1)}{2} = \frac{(n+1)n}{2}$

G. $A \in B \wedge B \subseteq C \Longrightarrow A \in C$. [redacted] False

H. There is no known algorithm to decide if a graph has a Hamiltonian cycle. True [redacted]

$\frac{40}{40}$

$\boxed{\pi \cdot \pi}$

$L^* = \bigcup_{n \geq 0} L^n$

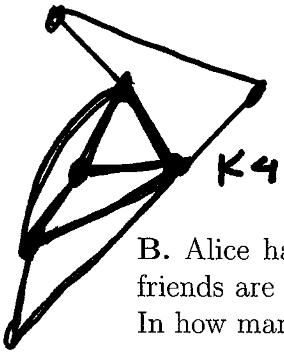
$L^0 = \{\varepsilon\}$

K_n

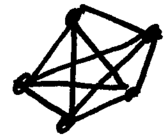
$$\boxed{\sum_{v \in V} \deg(v) = \underbrace{2m}_{\text{even}} \quad m = |E|}$$

2 Short Answer [70 points: 5 points each]

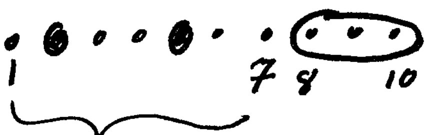
A. The graph K_{20} is a clique on 20 vertices. How many edges does it have?



$$C(20,2) = \frac{20\cdot 19}{2} = 190$$



B. Alice has 10 friends. She wants to invite five of them to dinner. Three of Alice's friends are inseparable, and so Alice must invite all three or them, or else none of them. In how many different ways can Alice choose the set of people to invite?



Doesn't Invite + Does Invite

$$C(7,5) + C(7,2) = 2\cdot C(7,2) = \frac{\cancel{2}\cdot 7\cdot 6}{\cancel{2}} = 42$$

$B(d,n)$

C. Draw a DFA (deterministic finite automaton) that recognizes the language of the regular expression $L = (aa)^* \vee ba$.

Regular language

$\{aa\}^* \cup \{ba\}$

$\{a\}, \emptyset,$ ~~$\{\varepsilon\}$~~

$|A| \le |B|$

$|A| = |B|$

D. Give a bijective function from N to Z (or indicate why none exists).

x	$f(x)$
0	0
1	1
2	-1
3	2
4	-2
⋮	⋮

$f: \mathbb{N} \to \mathbb{Z}$

$\mathbb{N} \subseteq \mathbb{Z}$

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

E. Define what it means for a function $f : \mathsf{N} \to \mathsf{R}$ to be $O(n)$. To get full credit you must give a mathematically precise definition.

$(\exists c)\ \ f(n) \le cn$ for all sufficiently large n

$(\exists c)(\exists N)(\forall n \ge N)(f(n) \le cn)$

F. $\lceil \lg 5! \rceil =$ $\underline{7}$

$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

$1, 2, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7$

120

$\lfloor \ \rfloor$

G. Use Euclid's algorithm to find integers x and y such that $19x + 35y = 1$.

$\gcd(a,b) =$ extended

$\gcd(a-b, b)$

$\gcd(a,b) = d$

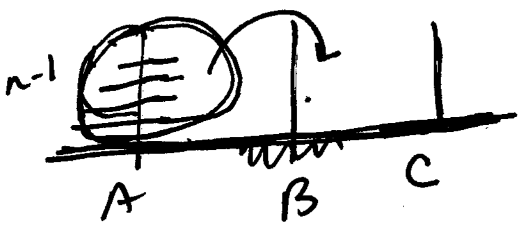
$\exists\, x, y.\ \ ax + by = d$

$\in \mathbb{Z}$

$35 - 19$

$\gcd(35,19) = \gcd(19,16)$
$= \gcd(16,3)$
$= \gcd(3,1) = \gcd(1,0) = 1$

H. Let $H_n =$ the minimum number of moves to solve the n-rings tower of Hanoi problem. Write a recurrence relation for H_n, where $n > 1$, just as we did in lecture 1.



$H_n = H_{n-1} + 1 + H_{n-1}$
$= 2H_{n-1} + 1$

$\boxed{T(n) = 2T(n-1) + 1}$

I. Solve the following recurrence relation: $T(n) = 2T(n/2) + 1$ for $n > 1$, and $T(n) = 0$ for $n = 1$. You may assume that n is a power of 2.

$T(n) = 2T(n/2) + 1$
$= 2[2T(n/4) + 1] + 1$
$= 2^2 T(\frac{n}{2^2}) + 1 + 2$
$= 2^3 T(\frac{n}{2^3}) + 1 + 2 + 2^2$
$\vdots$
$= 2^k \cancel{T(\frac{n}{2^k})} + \underbrace{1 + 2 + \cdots 2^{k-1}}$ $= 2^k - 1$

$k = \lg n$

$2^{\lg n} - 1 = n - 1$

$\underline{\in \Theta(n)}$

J. Define (formally!) what is a **graph** $G = (V, E)$. (Recall that, for us, graphs are undirected, finite, and have no loops or multiple edges.)

A graph $G = (V, E)$ is an ordered pair.

finite nonempty set ("vertices")

a set of 2-element subsets of V ("Edges")

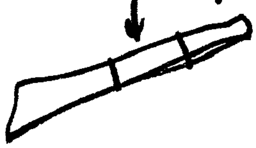
K. Compute the following number:

10's 10's 1's

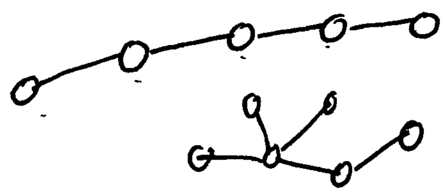
$$(348381 \times 9933829438 \times 445555481 \times 5349327413343) \bmod 10$$

$\mathbb{Z}_{10}$

4



L. What is the smallest number of edges that a connected graph on n nodes can have?



$n-1$

Trees = connected, acyclic graphs

n $n-1$ edges

M. The number of elements in the set

$$(\{1,2,3,4,5,6,7,8,9,10\} \times \{1,2,3,4,5,6,7,8,9,10\}) \cup (\{8,9,10,11,12\} \times \{8,9,10,11,12\})$$

(1,2)
(1,1)

is: 100 + 25 — 9

$|A \cup B| = |A| + |B| - |A \cap B|$

(8, 9, 10)
3^2

116

N. Consider the following definitions. The universe consists of various (types of) animals: lions, bears, whales, humans, mosquitoes, etc.

Fierce(x) — x is a fierce animal

LION — a constant, representing the animal which is a lion

$x > y$ — animal x is bigger than animal y

Now translate the following sentence into the predicate calculus. (Do not introduce any additional function symbols, constant symbols, or relation symbols.)

There is some non-fierce animal bigger than a lion.

$(\exists x)(\overline{\text{Fierce}(x)} \wedge x > \text{LION})$

3 A Little Proof [20 points]

Prove (from first principles—don't assume the un-countability of any particular set) that the set of all "infinite binary strings" (infinite sequences such as $001101\cdots$) is uncountable. (Remember that an infinite set is *uncountable* if it can not be put in one-to-one correspondence with $\mathbb{N}$.)

Assume that the set S of infininite binary string is countable. then $\exists$ an enumeration of it:

$S = \{x_1, x_2, \dots\}$

$x_1 = 0\,1\,1\,0\,0\,1\,0\,1\cdots$
$x_2 = 1\,0\,1\,0\,0\,1\,1\,0\cdots$
$x_3 = 0\,0\,0\,0\,0\,0\,0\,0\cdots$
$x_4 = 1\,0\,1\,0\,1\,0\,1\,0\cdots$
$x_5 = 0\,0\,0\,0\,1\,1\,1\,1\cdots$

Construct a ~~string~~ inf. binary string

$y = 11110\cdots$

by defining it i-th bit to by $\overline{x_i[i]}$

Claim: $y \notin \{x_1, x_2, \dots\}$ because $y \neq y_i$ for each $i \in \mathbb{N}$, it differs at bit posn i.

Contradiction.

4 A Little Counting [20 points]

In California's new "SuperLotto Plus" game, players select five different "regular numbers," each an integer between 1 and 47 (inclusive), and then they choose one "mega number," which is an integer between 1 and 27 (inclusive). (The Mega Number might be the same as one of the five regular numbers.) The game is played when the lottery officials select five random regular numbers and one random mega number. It costs $1 to play. Assume you buy one ticket.

A. The jackpot (which pays at least $7,000,000) is won for getting right all five regular numbers and the mega number. What is the probability of winning the jackpot?

5 : 1-47
1 : 1-27

$$\frac{1}{C(47,5)} \cdot \frac{1}{27} = 1/C(47,5)\cdot 27$$

B. You'll win $20,000 if you get four (not five) regular numbers plus the mega number. What is the probability of your winning this $20,000 prize? *(Hint: count the number of different tickets that will pay this prize.)*

$$\frac{5 \cdot 42}{C(47,5)\cdot 27}$$

43

{5, 8, 20, 30, 44}

16

5 · 42

C. You'll also win $20,000 if you get all five regular numbers but *not* the mega number. Which way (B or C) is more likely to happen, and why?

$$\frac{1}{\cancel{C(47,5)}}\left(\frac{26}{\cancel{27}}\right) \overset{?}{<} \frac{5\cdot 42}{\cancel{C(47,5)}\ \cancel{27}}$$

B is more likely.