# 1) A direct proof

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**Example 1**: $\sqrt{2}$ is irrational.

A number $x \in \mathbb{R}$ is <u>rational</u> if $x = a/b$ for some $a, b \in \mathbb{Z}$, $b \neq 0$.

A number $x \in \mathbb{R}$ is <u>irrational</u> if it is not rational.

Assume for contradiction that $\exists\, m, n \in \mathbb{Z}$, $n \neq 0$

and m and n have no common divisors.

$$\sqrt{2} = \frac{m}{n} \Rightarrow 2 = \frac{m^2}{n^2} \Rightarrow 2n^2 = m^2,$$ i.e., $m^2$ is even.

So m is even. $\exists\, k$ s.t. $2k = m$

$$2n^2 = (2k)^2 = 4k^2$$

$$n^2 = 2k^2$$

$n^2$ is even $\Rightarrow$ <u>$n$ is even.</u>

To show later: the square of a number being even $\Rightarrow$ the number is even

## 2) Clever cases

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**Example 2**: There are **irrational** numbers $a$ and $b$ such that $a^b$ is **rational**.

Is $x = \left(\sqrt{2}\right)^{\sqrt{2}}$ irrational?

<u>Case 1</u>: $\sqrt{2}^{\sqrt{2}}$ is rational. We're <u>done</u>.

<u>Case 2</u>: $\sqrt{2}^{\sqrt{2}}$ is irrational

$\underset{a}{\underset{\shortparallel}{\sqrt{2}^{\sqrt{2}}}}$ $b = \sqrt{2}$

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2 \in \mathbb{Q}$$

# 3) Exhaustive cases

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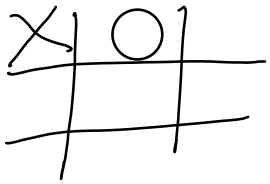
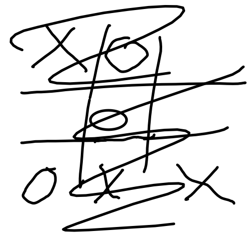
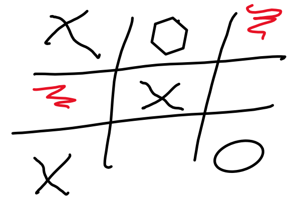
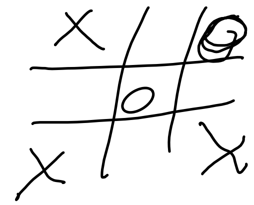
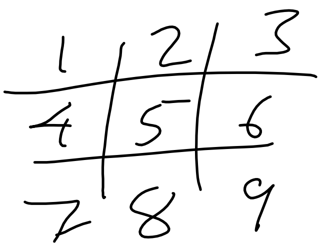
**Example 3**: In playing tic-tac-toe, if the first player moves to a corner then the second player must take the center, or else the first player can force a win.

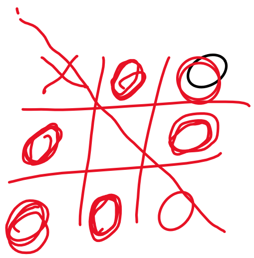


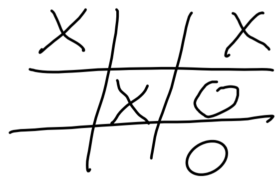


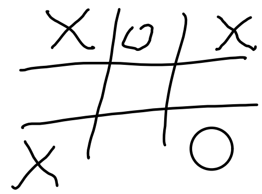










# 4) Add the right extra

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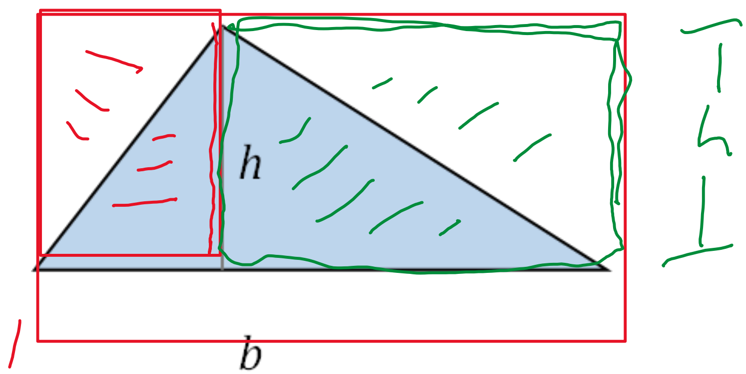
**Example 4**: The area of a triangle with three acute angles is $bh/2$ where $b$ is the triangle's base and $h$ is its height.



# 5) Algebra vs. pictures

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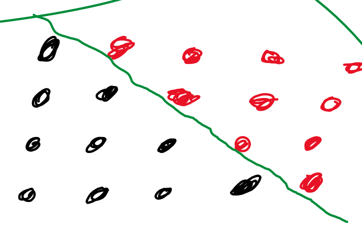
**Example 5:** $1 + 2 + \cdots + n = S = \frac{n(n+1)}{2}$

$n + (n-1) + \cdots + 1 = S$

$\underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n} = 2S$

$\frac{n(n+1)}{2} = S$

$n$



$n+1$

$\frac{n(n+1)}{2}$

# 6) More pictures

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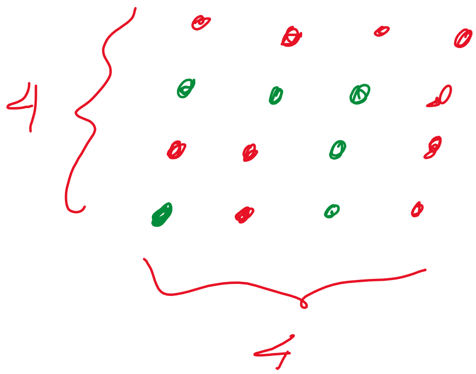
**Example 6**: $1 + 3 + 5 + \cdots + 2n - 1 = n^2$




| $n$ | $n^2$ | $1+3+\cdots+(2n-1)$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 4 | 4 |
| 3 | 9 | 9 |
| 4 | 16 | 16 |

# 7) Translate to the familiar

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**Example 7**:

20 random cards are placed in a row, all face **down**. A **move** consists of turning a face-down card face-up and turning over the card, if any, immediately to the **right**. Show that no matter what the choice of cards to turn, this sequence of moves **must** terminate.

D D D D D D

U U D

1 0 1 0 0 0 0 0 0

0 0 1 1 0 1 0 1 = a

0 0 1 1 1 0 0 1 = b = a''

1 1 1 1 0 1 0 1 = a'

# 8) Employ/invent the right notion

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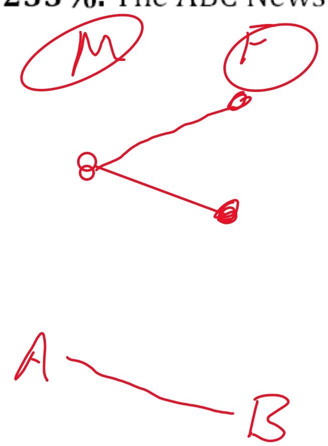
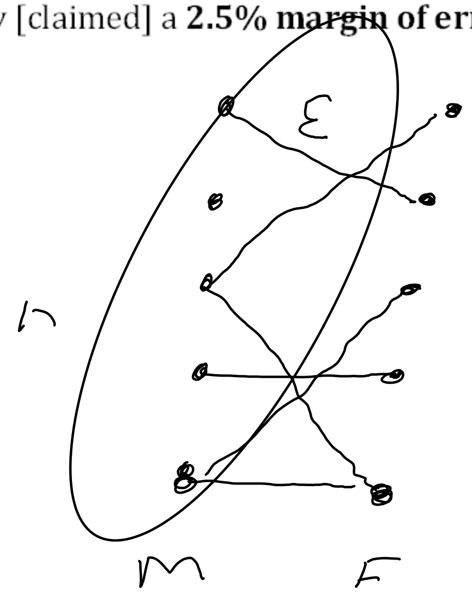
**Example 8**: [MIT book, chapter 11]

Who, on average, has more opposite-gender partners: men or women? ...

In one of the largest [studies], researchers from the University of Chicago interviewed a random sample of 2500 people over several years ... Their study, published in 1994, ... found that **men have on average 74% more opposite-gender partners than women**.

Other studies have found that the disparity is even larger. In particular, ABC News claimed that **the average man has 20 partners over his lifetime**, and **the average woman has 6**, for **a percentage disparity of 233%.** The ABC News study [claimed] a **2.5% margin of error**...




$$\frac{E}{n} = \frac{E}{n}$$

Bipartite graph

# Some truths about proofs

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1. Finding proofs is **not** mechanical; it is an **art**.
2. Mathematical discovery is **more than proofs**:
   guessing and discovering results.
   Only in a class setting do proofs come as “prepackaged” task
3. Don’t get so obsessed with rigor that you fail to develop and refine intuition – and to **err**.
4. Proofs **evolve.** They can be quite **dialectical.**
5. Intuition can be **lost** in a refined, succinct proof.
   Proofs are not “born” in such a manner
6. You can’t prove what doesn’t **make sense** to you.
   Don’t even try to prove something until you get to the point of the language and claim making sense.