ECS-20 Section A (MWF) – Fall 2021

First name: ____________ **LAST NAME:** ____________

Room # ______ Seat # ______

"I attest that I have in no manner cheated on this exam."

Signature: ____________ **SID:** ____________

Instructions

(1) The exam has five numbered pages, not including this one.
(2) You may not sit next to anyone you know.
(3) No calculators, phones, or smartwatches. Devices must be powered off and put in your bag.
(4) You may have one page of notes, 8.5×11, printed on one-side. Put your name on it and turn it in with your exam.
(5) Read each question *carefully* and write each answer *neatly*.
(6) If you don't understand some question or notation, you can ask.
(7) On the True/False, if you don't know, guess. Your score, prior to rounding to the nearest natural number, will be linearly proportional to the number of correct answers.
(8) When an answer is a number, you can name it explicitly or provide an expression involving basic arithmetic, factorials, $C(n, k)$, or $P(n, k)$.

On page	you got	out of
1–2		120
3		60
4		60
5		60
Σ		300

1 True / False

120 points

Darken the **correct** answer. If you don't know, **guess**.

1. If P is False then $P \to Q$ is True. **True** **False**
2. The XOR of 80 Boolean variables will be 1 if half of them are 1. **True** **False**
3. $P \to (Q \to R) \equiv (P \to Q) \to R$ **True** **False**
4. $\models (X_1 \leftrightarrow X_2)(X_2 \leftrightarrow X_3) \to (X_1 \leftrightarrow X_3)$. **True** **False**
5. It is possible to realize a NOR gate using only XOR gates. **True** **False**
6. If a, b, and x are bits and $a \oplus x = b$ then $x = a \oplus b$. **True** **False**
7. If ϕ is a CNF formula then $(\neg\phi)$ is a DNF formula. **True** **False**
8. $\overline{(\exists n)(P(n) \wedge P(n+1) \wedge n > 2)} \;\equiv\; (\forall n)(\overline{P(n)} \vee \overline{P(n+1)} \vee n \leq 2)$ **True** **False**
9. For any sets A, B, and C, if $A \times B = A \times C$ then $B = C$. **True** **False**
10. The two's complement representation of -2 as an 8-bit number is `11111110`. **True** **False**
11. $A \subseteq \mathcal{P}(A)$ for any set A. **True** **False**
12. The set of 32-bit numbers forms a group under addition mod 2^{32}. **True** **False**
13. The group S_{10} of permutations on $\{1, 2, \ldots, 10\}$ has 2^{10} points. **True** **False**
14. For any relation $\sim \subseteq X \times X$, if $a \sim b$ and $b \sim c$ then $a \sim c$. **True** **False**
15. There is an injective function $f\colon \mathbb{N} \to \mathbb{R}$. **True** **False**
16. There is an injective function $f\colon \mathbb{R} \to \mathbb{N}$. **True** **False**
17. If $f\colon A \to B$ and $g\colon B \to C$ are bijective then $g \circ f\colon A \to C$ is bijective. **True** **False**

18. If A is a finite set then $|A \times A| = |A|^2$ **True** **False**

19. $f(n) = 2n \bmod 5$ is a bijection on $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. **True** **False**

20. Every language $L \subseteq \{0, 1\}^*$ can be decided by some computer program. **True** **False**

21. If L is a finite language then $|L \circ L| = |L|^2$. **True** **False**

22. If a language L contains a string of length 5 then L^* is infinite. **True** **False**

23. $\{00, 01, 10, 11\}^* = \{0, 1\}^*$ **True** **False**

24. There is a BNF[1] description of any finite language. **True** **False**

25. Algorithm D runs in 2^{80} steps, each taking 1 machine cycle. You need to run it once a day. Doing so should be feasible on a fast, recent computer. **True** **False**

26. If $f(n) \in O(n)$ and $g(n) \in O(n)$ then $(f(n) + g(n) + \log(n)) \in O(n)$. **True** **False**

27. $\sum_{i=1}^{n} i^2 \in \Theta(n^2)$. **True** **False**

28. $C(n, 2) \in \Theta(n^2)$. **True** **False**

29. $P(n, k) = P(n, n-k)$ for all $1 \leq k \leq n$. **True** **False**

30. There are more than 100 ways to rearrange the letters of `zebra`. **True** **False**

31. To talk about the probability of being dealt various poker hands we use a probability space (S, μ) with a sample space S consisting of $|S| = 52$ cards. **True** **False**

32. For any blockcipher $E\colon \{\mathtt{0},\mathtt{1}\}^{256} \times \{\mathtt{0},\mathtt{1}\}^{128} \to \{\mathtt{0},\mathtt{1}\}^{128}$, $E(K, X) = E(K', X')$ implies $(K, X) = (K', X')$. **True** **False**

33. Graph K_{10}, the clique of size 10, is Eulerian.[2] **True** **False**

34. Graph K_{10}, the clique of size 10, is 2-colorable (bipartite). **True** **False**

[1] Backus Normal form, as in PS #3, problem 5.

[2] Recall that K_n has n vertices and an edge between each pair of them; and that a graph is **Eulerian** if there is a cycle that passes through each of its edges once and only once.

2 Short Answer

180 points — 10 points each

If you see a box, make sure your final answer lives in it. Don't worry; it won't feel claustrophobic.

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1. $\gcd(860, 200) =$ ☐

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2. How many (positive) **divisors** $d(n)$ does $300125 = 5^3\,7^4$ have? ☐

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3. Explicitly name the elements: $\{1, 2\} \times \{3, 4\} =$ $\{\qquad\}$

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4. How many **functions** are there from $\{0, 1\}^3$ to $\{1, 2, 3\}$? ☐

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5. Neatly write a **truth table** for the Boolean function $A \to (B \to C)$. Order the rows in the conventional way. Use 0 and 1 to represent false and true.

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6. Draw a **circuit** that computes the XOR function, $y = a \oplus b$, limiting yourself to only AND, OR, and NOT gates. Don't use more gates than necessary. Note: If you forgot the symbol for a gate just draw a box and label it AND, OR, or NOT.

7. Suppose I've defined a function $f : \mathbb{N} \to \mathbb{N}$ and I want to show that $f(n) \leq 10n$ for all $n \geq 5$. To do this by induction it suffices to prove that

   [ ] (the basis) and that

   [ ] (the inductive step).

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8. A woman has ten close friends. In how many ways can she invite five of them to dinner if two of them are inseparable and must either both be invited, or neither one invited?

   [ ]

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9. An urn contains 15 red balls and 10 white balls. Five random balls are removed. What is the probability that all of them are red?

   [ ]

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10. What **exactly** would you enter in LaTeX to typeset the follow: Let $f(n) = n^2 \log B_n$

    [ ]

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11. Consider the recurrence: $T(n) = T(n/2) + 1$. Assume $T(n) = 1$ when $n \leq 1$. Solve the recurrence to within a $\Theta(\cdot)$ bound: $T(n) \in$ [ ] .

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12. Define the equivalence relation $\sim \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ by: $a \sim b$ if a and b, written as decimal numbers, have the same first (leftmost) digit. This equivalence relation has [ ] equivalence classes and the three smallest elements in one of them are: [ ]

    Note: $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

13. How many ways can you paint the **edges** of K_{10} with each edge red or blue? [ ]

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14. Where did you encounter the following? [ ]

    Twenty random cards are placed in a row, all face-down. A *move* consists of turning a face-down card face-up, and turning over the card immediately to the right. Show that no matter what the choice of cards to turn, this sequence of moves **must** terminate.

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15. *Diagonalization.* A **superstring** is an infinite sequence of bits $A = A[1]\,A[2]\,A[3]\cdots$ of bits. (Write $A[i]$ for the ith bit of a superstring A.) Let's prove that the set $\mathbb{S}$ of all superstrings is **uncountable**.

    Suppose for contradiction that $\mathbb{S}$ were countable. Then there would be some enumeration of it, a list $X_1, X_2, X_3, \ldots$ that includes all the superstrings.
    Consider the superstring $B = B[1]\,B[2]\,B[3]\cdots$ defined by saying that that

    $B[i] =$ [ ] .

    Is B in the list $X_1, X_2, X_3, \ldots,$? If so then $B = X_j$ for some particular $j \in \mathbb{N}$. Yet it can't be the case that $B = X_j$ because

    [ ] .

    That B is **not** in $X_1, X_2, X_3, \ldots$ contradicts that list containing all superstrings.

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16. How many strings can you get by rearranging the characters of `aardwolf`? [ ]

    Note: if you flip the first two letters, say, it's not a new string.

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17. In the "Monty Hall" ("Let's Make a Deal") problem from our final lecture, what is a contestant's probability to select the correct door assuming she always does switch doors when the host offers her the opportunity? [ ]

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18. *Complete the following proof that some positive multiple of 159 has digits that are all 0s and 1s. Be succinct.*

    For $i \in [1..160]$, let $A_i = \overbrace{111\cdots111}^{i}$ and let $a_i = A_i \bmod 159$.
    By the