Midterm Solutions

Graded out of 118 points: 5 points per problem except: $\max(0, 6m - 33)$ for m correct T/F answers on problems 1–11 (corresponds to +3 for correct T/F answers; -3 for incorrect T/F answers; some response expected on each question); 10 points for problem 26.

1. The formula $\phi = (P \vee Q \vee R)$ has exactly *seven* satisfying assignments. (Assume a universe of variables of $\mathcal{U} = \{P, Q, R\}$). **True**
2. It is possible to realize a NAND gate using OR and AND gates. **False**
3. It is possible to realize an XOR gate using NAND gates. **True**
4. We described in class how every formula ϕ is either *complete* or *sound*. **False**
5. There is a formula ϕ such that ϕ is satisfiable **and** $\neg\phi$ is satisfiable. **True**
6. If A, B, and X are bits, and $X \oplus A = B$, then $X = A \oplus B$. **True**
7. For sets X, Y, and C, if $X \cup C = Y \cup C$ then $X = Y$. **False**
8. For every set A it is the case that $\emptyset \subseteq A$. **True**
9. Regular expressions $(a \cup b)^*$ and $(aa^* \cup b)^*$ denote the same language. **True**
10. Strings under concatenation form a group. **False**
11. Let A and B be sets of strings. Then $A \times B = B \times A$. **False**
12. Let A and B be finite sets. Then $|AB| = |A| \cdot |B|$ *This was supposed to say finite sets* ***of strings****, which would make the answer false. But, not having said that, you might have assumed I meant Cartesian product in writing AB, which would make the answer true. So I didn't grade the problem.*
13. Remember the **Towers of Hanoi** problem, where we have n rings on one of three pegs. Using the recursive algorithm described in class, the number of moves T_n needed to transfer these n rings to a different peg is given by the recurrence relation: $T_0 = \boxed{0}$ and $T_n = \boxed{2T_{n-1} + 1}$ for all $n \geq 1$.
14. Starting at 0, count in **binary** (base-2): $\boxed{0}\ \boxed{1}\ \boxed{10}\ \boxed{11}\ \boxed{100}$.
15. A **truth table** for $Y = (A \leftrightarrow B) \wedge (B \leftrightarrow C) \wedge (C \leftrightarrow D)$ has 16 rows. It has four columns to specify the input (A, B, C, D) and one column to specify the output (Y). Of the 16 bits that occur in the output, $\boxed{14}$ are zero (0) and $\boxed{2}$ are one (1).
16. Write a **disjunctive normal form** (**DNF**) formula whose truth table is given below. Your formula should be the **or** of terms where each term is the **and** of variables or their complements: $\boxed{\overline{p}\,\overline{q}\,r \vee p\,q\,\overline{r}}$
17. **Negate** and simplify the following formula. Your answer should only use addition, exponentiation, $\{\wedge, \vee\}$, and $\{<, \leq, =, \neq, >, \geq\}$.

    $\neg((\exists n)(\exists a)(\exists b)(\exists c)(a^n + b^n = c^n \wedge n \geq 3))$ is equivalent to

    $(\forall n)(\forall a)(\forall b)(\forall c)(a^n + b^n \neq c^n \vee n < 3))$

    The picture is of **Andrew Wiles** (1 EC point if you knew his name) — see him on YouTube!

18. **Translate** the following English sentence into a logical formula:

    Some cats can dance—but no such cat can also fetch the morning paper.

    The universe $\mathcal{U}$ is "animals." Use predicates of: $C(x)$ for x is cat; $D(x)$ for x can dance; and $F(x)$ for x can fetch the morning paper.

    $(\exists x)(C(x) \wedge D(x)) \wedge \neg(\exists x)(C(x) \wedge D(x) \wedge F(x))$

    or

    $(\exists x)(C(x) \wedge D(x)) \wedge (\forall x)(C(x) \wedge D(x) \rightarrow \neg F(x))$

19. Express the following equality in **compact mathematical notation**: *The sum of the first 100 positive integers is 5050.*

$$\sum_{i=1}^{100} i = 5050$$

20. Express the following as a sentential formula (no quantifiers) in **compact mathematical notation**: *At least two of the boolean variables $X_1, \ldots, X_{100}$ are true.*

$$\bigvee_{1 \leq i < j \leq 100} X_i X_j$$

21. We used the **compactness theorem** of sentential logic to show that *what* is true about tiling the plane using a specified set of tile types?

    If the plane is not tileable then some finite portion of the plane is already not tileable.

22. To use **mathematical induction** to prove that a proposition $B(n)$ is true for all numbers $n \geq 72$, show that $\boxed{B(72)}$ and $\boxed{B(n) \rightarrow B(n+1) \text{ for all } n \geq 72}$.

23. Explicitly specify the **power set** of the given set: $\mathcal{P}(\{\text{big, ghost}\}) = \boxed{\{\emptyset,\ \{\text{big}\},\ \{\text{ghost}\},\ \{\text{big,ghost}\}\}}$.

24. Let BYTES be the set of 8-bit strings. We defined two addition operations on BYTES that made this set into a *group*: **bitwise-XOR** ($\oplus$) (below left) and (carryless) **computer addition** ($+$) (below right). Add the numbers using each operation.

$$\begin{array}{c|cccccccc} & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ \oplus & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ \hline & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array} \qquad \begin{array}{c|cccccccc} & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ + & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ \hline & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$

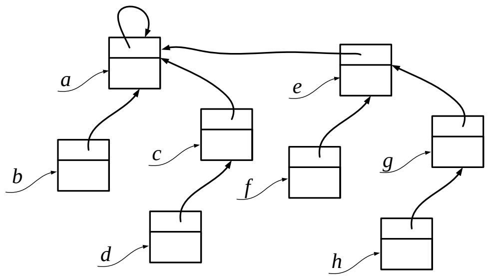
25. Let $R \subseteq \{a, b\}^* \times \{a, b\}^*$ be the **equivalence relation** defined by $x\,R\,y$ iff the string $|x| = |y|$. Explicitly list the elements of $[aa]$, the block (equivalence class) containing aa.

$$[aa] = \boxed{\{aa, ab, ba, bb\}}$$

26. In the Kingdom of Konfusion, coins come in nimes (9¢) and dimes (10¢). Prove that it's possible to make any integral number $n \geq 72$ of cents using only nimes and dimes.

    See the solution on PS4.

27. If you use the **UNION/FIND** data structure with union-by-rank and collapsing-find, what do you return—and what side effects do you cause—if you call FIND(h) with the following data structure?



FIND(h) will return a.

With collapsing find, we will also adjust the parent pointer of the objects pointed to by h and g to point to a.

28. Write a shortest **regular expression** for the language L that is the set of all binary strings $x \in \{0,1\}^*$ whose length is divisible by three.

$((0 \cup 1)(0 \cup 1)(0 \cup 1))^*$

29. *Extra credit.* What's an **aardwolf's** favorite food?

$\boxed{\text{They like } \textbf{termites}\text{, same as me.}}$