ECS 20 – Winter 2022 – P. Rogaway

Counting 1

Today:

- **Let's count!**

First, the "theory"

2^n = Number of subsets of n items
= number of n-bit binary strings
= number of ways to paint n items with 2 different colors.

d^n = Number of length-n strings over an alphabet of d character
= number of ways to paint n items with d different colors.

$n!$ = Number of ways to arrange n different items
= Number of ways to order $\{1,2,\ldots,n\}$

$P(n, k)$ = The number of ways to arrange k items drawn, without replacement, from a universe of n items
= Number of ways to fill k bins, one item per bin, from a universe $\{1,\ldots,n\}$
= $n(n-1)\ldots(n-k+1)$
= $n!/(n-k)!$

No replacement; an item, once used, is gone.

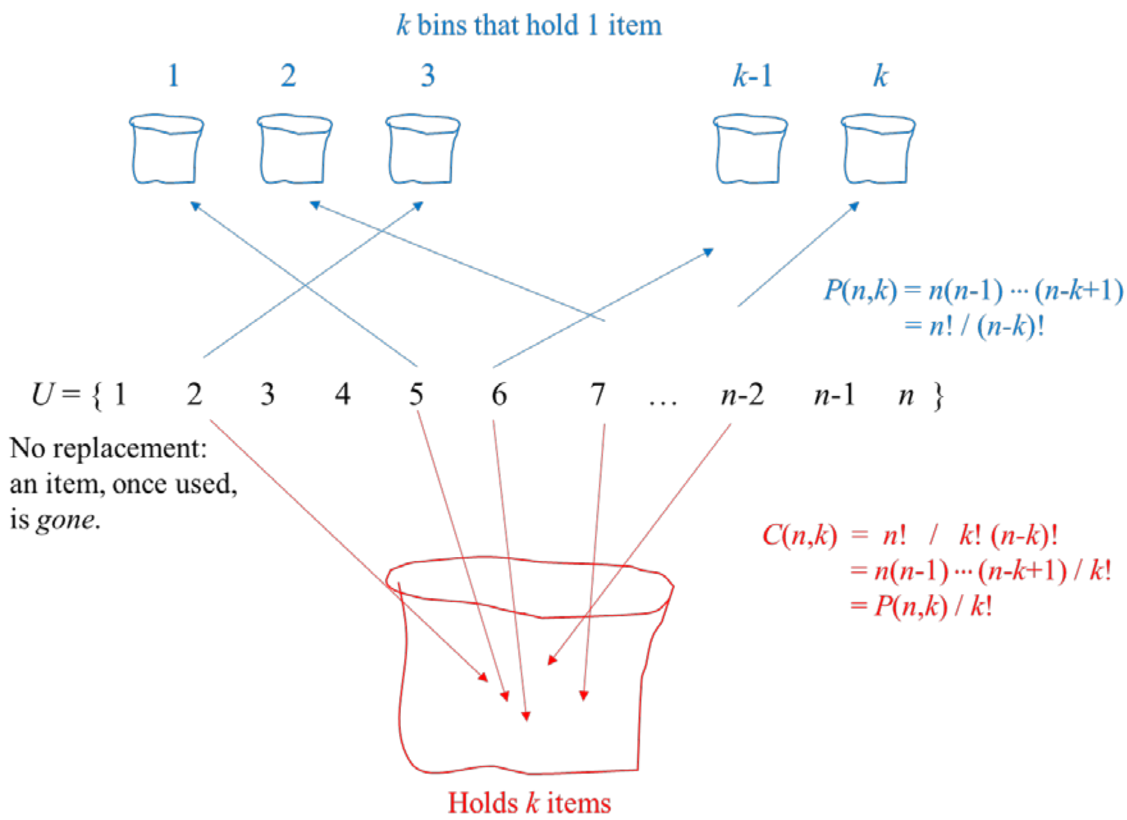
$C(n, k)$ = Number of ways to fill a bin with k items from a universe $\{1,\ldots,n\}$
= number of k-element subsets from a set of n different items
= $n!\ /\ k!(n-k)!$
= $P(n, k)/k!$

No replacement; an item, once used, is gone.

Supported by Google's search-line calculator as in "100 choose 50"

Alternate notation: $\binom{n}{k}$

$C(n, 2)$ = Number of 2-element subsets from an n-element set
= number of k-element subsets from a set of n different items
= $n(n\text{-}1)/2$



product rule = if event A can occur in a ways and, independent of this,
event B can occur in b ways
then the number of combinations of ways
for A and B to occur is ab.

- Really just a statement that $|A \times B| = |A|\,|B|$ for finite A, B.

sum rule = if event A can occur in a ways and
event B can occur in b ways,
but both events cannot occur together,
then the number of ways for A **or** B to occur is $a+b$.

- Really just a statement that $|A \cup B| = |A|+|B|$ for disjoint A, B.

Inclusion/exclusion counting:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

And generalizations, like

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Reminder: $\log(n!) \approx n \log n$

Example counting exercises

Please calculate values explicitly to the point of getting out numbers – I like to see actual numbers.

1. How many ways can a blue, white, and red ball be put into 10 different bins? Assume no bin can contain two balls.

   Answer: 10*9*8 = $P(10,3)$ = 720

2. License plates in Nebraska are 3 distinct letters (A-Z, but not O), followed by 3 distinct decimal digits. How many possible license plates are there?

   Answer: 25*24*23*10*9*8 = $P(25,3)$ $P(10,3)$ = 9,936,000

3. How many different ways a salesman travel among n cities, where he starts in city 1 and visits each other city once and only once before returning to city 1.

   Answer: $(n-1)!$

4. How many ways can you select a president, vice president, and treasurer in a club of 30 people?

   Answer: $P(30,3)$ = 24,360

5. How many way can you form Male-Female dance partners if there are 12 women and 8 men. Assume each man is partnered with some woman (4 women go un-partnered).

Answer: $P(12,8)$ = 19,958,400

6. How many ways you position 7 people in a circle?

Answer: 6! = 720

7. A man, a woman, a boy, a girl, a dog, and a cat are walking single-file down the road.

a. How many ways can this happen?
Answer: 6! = 720

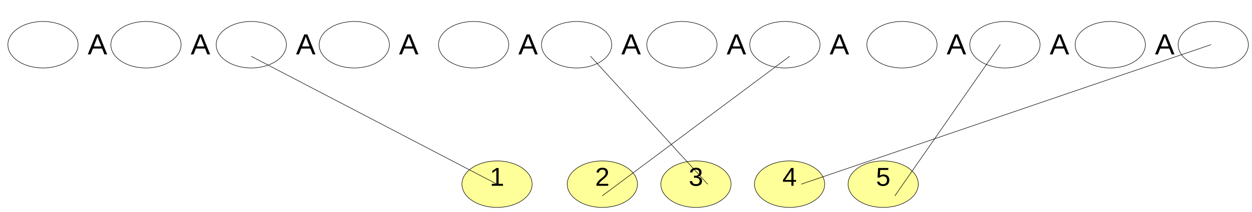
b. How many ways if the dog comes first?
Answer: 5! = 120

c. How many ways if the dog immediately follows the boy?
Answer: 5! = 120

d. How many ways if the dog (and only the dog) is immediately between the man and the boy.
Answer: 2*4! = 48 (form a man-dog-boy or a boy-dog-man combo)
(so walking down the street is a woman, a girl, a cat, and a man-dog-boy (4!)
or, walking down the street is a woman, a girl, a cat, and a boy-dog-man (4!)

8. In how many ways can 10 adults and 5 children be positioned in a line so that no two children are next to each other? (they fight)

Answer: $10!*P(11,5) = 10!\ 11!\ /\ 6! = 201{,}180{,}672{,}000 \approx 10^{11.3}$



9. How many arrangements are there of the letters A..Z such that there are exactly 10 letters between the A and the Z?

Answer: $15!*P(24,10)*2 = 24!*30 \approx 1.86*10^{25}$

(reasoning: after selecting the AxxxxxxxxxxZ block, treat it as atomic and rearrange it with the 14 remaining letters in any of 15! ways. Double to account for both the AxxxxxxxxxxZ and ZxxxxxxxxxxA possibilities.)

10. You take a group of four people to a Chinese restaurant that has 100 different dishes. All food will be shared among the four of you. How many ways can you order 4 different dishes?

Answer: $C(100,4) = 100*99*98*97 / (4*3*2*1) = 3{,}921{,}225$