ECS 20 – Winter 2022 – P. Rogaway

Lecture 17: Counting 2

Today:

- **Review**
- **More counting**
- **Naïve probability**

Review

2^n = Number of subsets of n items
= number of n-bit binary strings
= number of ways to paint n items with 2 different colors.

d^n = Number of length-n strings over an alphabet of d character
= number of ways to paint n items with d different colors.

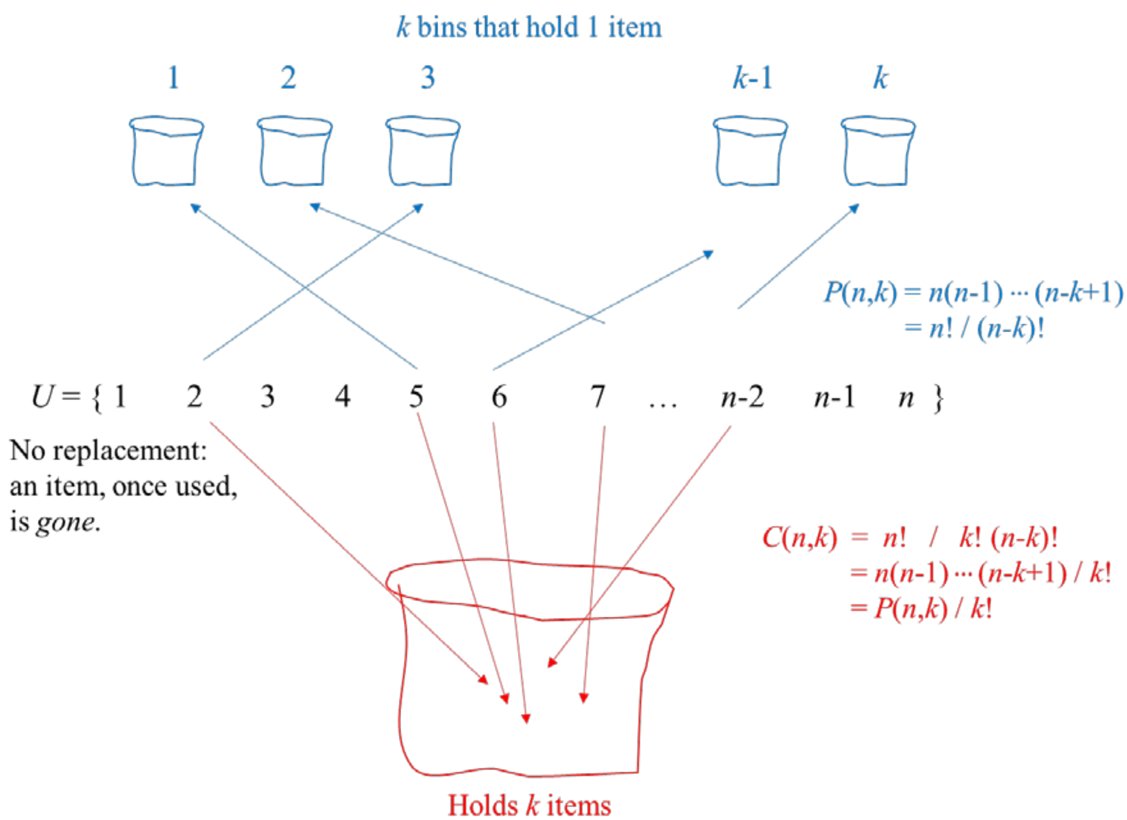
$n!$ = Number of ways to arrange n different items
= Number of ways to order $\{1,2,\ldots,n\}$

$P(n, k)$ = The number of ways to arrange k items drawn, without replacement, from a universe of n items
= Number of ways to fill k bins, one item per bin, from a universe $\{1,\ldots,n\}$
= $n(n-1) \ldots (n-k+1)$
= $n!/(n-k)!$
No replacement; an item, once used, is gone.

$C(n, k)$ = Number of ways to fill a bin with k items from a universe $\{1,\ldots,n\}$
= number of k-element subsets from a set of n different items
= $n! \;\; / \;\; k!(n-k)!$
= $P(n, k)/k!$
No replacement; an item, once used, is gone.
Supported by Google's search-line calculator as in "100 choose 50"

Alternate notation: $\binom{n}{k}$

$C(n, 2)$ = Number of 2-element subsets from an n-element set
= number of k-element subsets from a set of n different items
= $n(n$-$1)/2$



product rule = if event A can occur in a ways and, independent of this,
event B can occur in b ways
then the number of combinations of ways
for A and B to occur is ab.

- Really just a statement that $|A \times B| = |A|\,|B|$ for finite A, B.

sum rule = if event A can occur in a ways and
event B can occur in b ways,
but both events cannot occur together,
then the number of ways for A **or** B to occur is a+b.

- Really just a statement that $|A \cup B| = |A|+|B|$ for disjoint A, B.

Inclusion/exclusion counting:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

And generalizations, like

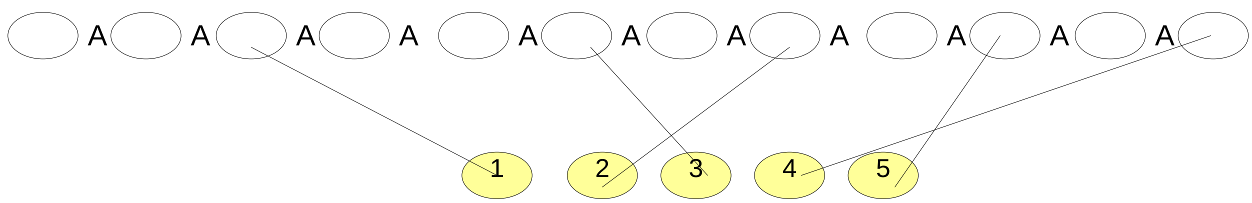
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Reminder: $\log(n!) \approx n \log n$

Review

Problem 8. In how many ways can 10 adults and 5 children be positioned in a line so that no two children are next to each other?

Answer: $10!*P(11,5) = 10!\ 11!\ /\ 6! = 201{,}180{,}672{,}000 \approx 10^{11.3}$



More Examples

Problem A. Ten members of a club line up for a photograph. The club has **one president**, **one VP**, **one secretary**, and **one treasurer**.

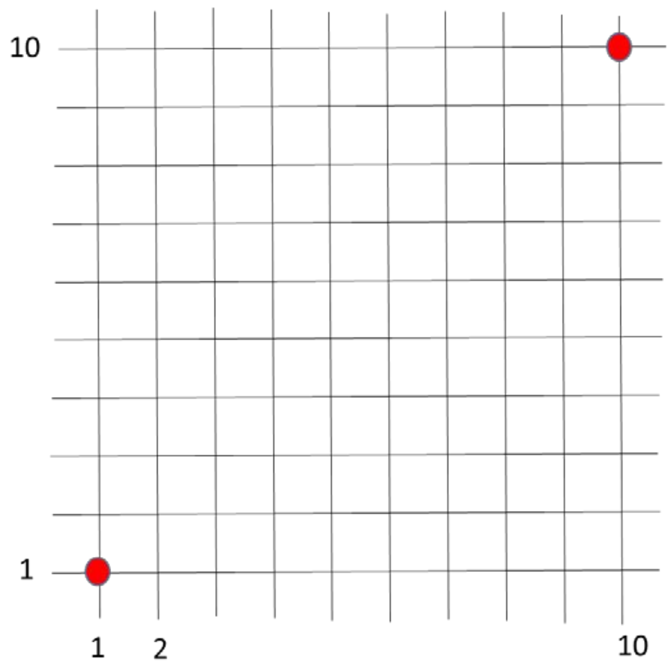
- How many ways are there to line up the ten people?
  10!
- How many ways are there to line up the ten people if the VP must be beside the president in the photo?
  $2 \cdot 9!$
- How many ways are there to line up the ten people if the president must be next to the secretary and the VP must be next to the treasurer?
  $2 \cdot 2 \cdot 8!$

Problem B. You toss a coin **8** times. How many ways can the coin tosses land? How many ways with **5** heads total?

Answer: $C(8,5) = 56$

Note this is $C(8,5)=C(8,3)$.
In general, $C(n, k) = C(n, n\text{-}k)$.

Also notice that $2^8 = C(8,0) + C(8,1) + \ldots + C(8,8)$.).
In general, $2^n = C(n,0) + C(n,1) + \ldots + C(n, n)$



Problem B. How many paths take you from (1,1) to (10,10) where each step take you either north or east?

EENNENNEEENNNENEEN

C(18,9)

Problem D. How many 6-element subsets are there of the letters A ... Z?
$C(26,6) = 230{,}230$

How many 2-element subsets are there of the letters A ... Z ?
$C(26,2) = 26*25/2 = 325$. In general, $C(n,2) = n(n-1)/2$

Are there more 2-element subsets or 24-element subsets?
The same.

Are there more 2-element subsets or 3-element subsets?
More 3-element subsets. $(n\text{-}2)/3 = 8$ times as many.

How many subsets all together are there of are there of A … Z ?

$2^{26} = 67{,}108{,}864$

Problem E. An urn contains 15 red, distinctly numbered balls, and 10 blue, distinctly numbered balls.
5 balls are removed.

(A) How many different samples are possible?
Answer: $C(25,5) = 53{,}130$

(B) How many samples contain only red balls?
Answer: $C(15,5) = 3003$.

(B') So what is the **probability** that a random sample will contain only red balls?
Answer: 3003 / 53,130 ≈ 0.05652 (05.652 %) (a little more than about 1 in 18)

(C) How many samples contains 3 red balls and 2 white balls?
Answer: $C(15,3) * C(10,2) = 20{,}475$

(C') So what's the probability that a random sample will contain 3 red balls and two blue ball?
Answer: 20,475 / 53,130 ≈ 0.3854 (38.54%)

Problem F. How many numbers are there between 1 and 1000 have are **not** divisible by 3, 5, or 7

A_3 = numbers in [1..1000] that **are** divisible by 3. $|A_3|=333$
A_5 = numbers in [1..1000] that **are** divisible by 5. $|A_5|=200$
A_7 = numbers in [1..1000] that **are** divisible by 7. $|A_7|=\lfloor 1000/7 \rfloor=142$

$A_{3,5}$ = numbers in [1..1000] that are divisible by 3 & 5. $|A_{3,5}| = \lfloor 1000/15 \rfloor = 66$
$A_{5,7}$ = numbers in [1..1000] that are divisible by 5 & 7. $|A_{3,5}| = \lfloor 1000/35 \rfloor = 28$
$A_{3,7}$ = numbers in [1..1000] that are divisible by 3 & 7. $|A_{3,5}| = \lfloor 1000/21 \rfloor = 47$

$A_{3,5,7}$ = nums in [1..1000] that are divisible by 3&5&7. $|A_{3,5,7}| = \lfloor 1000/3*5*7 \rfloor = 9$

So answer, by inclusion/exclusion, is $1000 - 333 - 200 - 142 + 66 + 28 + 47 - 9 =$ **457**

Problem G. Poker. Deck of 52 cards, these having 13 "values" and 4 suits. 5 cards are dealt. We are interested in the probability of being dealt certain kinds of hands.

royal flush = 10JQKA of one suit.
straight flush = five consecutive cards: 2345, , ..., , 10JQKA in any suit.
four of a kind = four cards of one value (e.g., all four 9's)
full house = 3 cards of one value, 2 cards of another value. (Eg, 3xA, 2x4).
flush = five cards of a single suit
three of a kind = 3 cards of one value, a fourth card of a different value, and a fifth card of a third value
two pairs = two cards of one value, two more cards of a second value, and the remaining card of a third value
one pair = two cards of one value, but not classified above

a) How many poker hands are there?

Answer: $C(52,5)=2,598,960$

b) How many poker hands are full houses?

Answer: A full house can be **partially** identified by a pair, like (J,8), where the first component of the pair is what you have **three** of, the second component is what you have **two** of. So there are $P(13,2)=13*12$ such pairs.

For each there are $C(4,3)=4$ ways to choose the first component, and $C(4,2)=6$ ways to choose the second component. So all together there are **13*12*4*6=3,744 possible full houses.**

c) What's the probability of being dealt a full house?

3,744/2,598,960 $\approx$ 0 .001441 $\approx$ 0.14%

P[FullHouse] $\approx$ .001441

The **probability** of an event is a real number between 0 and 1 (inclusive). If asked what's the probability of something, don't answer with a "percent", and don't answer with something outside of [0,1]. When we give something in "percent's", we are giving a probability multiplied by 100.

d) How many poker hands are **two pairs?**

Answer: We can partially identify two pairs as in {J, 8}. Note that now the pair is now **unordered**. There are $C(13,2)$ such sets. For each there are $C(4,2)$ ways to choose the larger card and $C(4,2)$ ways to choose the smaller card. There are now 52 - 8 =44 remaining cards one can choose as the fifth card (to avoid a full house, there are 8 "forbidden" cards). So the total is

$$C(13,2)*C(4,2)*C(4,2)*44 = 123{,}552.$$

e) What is the probability of being dealt two pairs?

$C(13,2)*C(4,2)*C(4,2)*44 / C(52,5) =$ 123,552/2,598,960
$\approx 0.047539 \approx 4.75\%$

P[TwoPairs] 0.047539

Problem H. How many different passwords are there that contain only digits and lower-case letters and satisfy the given restrictions?

(a) Length is 6 and the password must contain at least one digit.

All length-6 passwords – Those with only letters
$= 36^6 - 26^6 = 1867866560 \approx 2^{31}$

(a) Length is 6 password that contain **at least one digit** and **at least one letter**.

All length-6 passwords – Those with only letters - Those with only digits
$= 36^6 – 26^6 – 10^6 = 1866866560 \approx 2^{31}$

Problem I. A 5-card hand is drawn from a deck of standard playing cards.

(a) How many **5-card hands** have **at least one club?**

Total # Of Deals – Number Of Deals With No Clubs
= C(52,5) – C(39,5) = 2023203

(b) How many 5-card hands have **at least two cards** with the same rank?

Total # Of Deals – # of Deal where All 5 cards have different ranks
$= C(52,5) – C(13,5)\ 4^5 = 1281072$