ECS 20 – Winter 2022 – P. Rogaway

Lecture 18: Probability

Today:

- **Examples & Foundations (interspersed)**

Examples

Problem 1.

You role a pair of fair dice. What's the probability of getting an 8?

Event $E = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$
$P(E) = 5/36$

Be careful: $P(E) = |E|/|S|$ only if we are assuming the uniform distribution.

Foundations

Def: A (finite) **probability space** (S, p) is

- a finite set S (*the sample space*) and
- a function p: $S \to [0,1]$ (the *probability measure*) such that

$$\sum_{x \in S} p(x) = 1$$

(Alternative notation: Ω for S; μ or P for p)

In general, whenever you hear *probability* make sure that you are clear **what** is the probability space is: what is the **sample space** and what is the **probability measure** on it.

Def: An **outcome** is a point in S.

Def: An **event** $A \subseteq S$ is a *subset* of S.

Def: Let A be an event of probability space (S, p).

$$P(A) = \sum_{a \in A} p(a)$$ // The notation Pr is common, too

Propositions:

- $P(\emptyset) = 0$ and $P(S) = 1$ (by definition)
- $P(A) + P(A^c) = 1$, or $P(A) = 1 - P(A^c)$
- If A and B are **disjoint** events (that is, disjoint sets) then $P(A \cup B) = P(A) + P(B)$
- **Sum bound:**
  $$P(A \cup B) \leq P(A) + P(B)$$
- **Principle of inclusion-exclusion:**
  $$P(A \cup B) = P(A) + P(B) - P(\mathrm{A}\cap B)$$

Def: The **uniform** distribution is the one where $P(a) = 1/|S|$ —i.e., all points are equiprobable.

Def: Events A and B are **independent** if $P(A\cap B) = P(A)\, P(B)$.

Let's play: *identify the probability space.*

- You roll a fair die one time:
  $S = \{1,2,3,4,5,6\}$
  $p(1) = p(2) = \ldots = p(6)=1/6$

  "you roll an even number" is an event.
  Event is $A = \{2,4,6\}$. $P(A) = 3 * (1/6) = 1/2$.

- You roll a pair of dice.
  $S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$
  $p((a, b)) = 1/36$ for all $(a, b) \in \mathrm{S}$

note: The singular of dice, the students assure me, is *die*.
Just as the singular of *mice* and *mie*, I suppose.

Illustrate independence.

P(left die even and right die even) = *P*(left die even) *P*(right die even)
= (1/2) (1/2) =1/4

Often we need our combinatorics skill to answer probability questions.

Problem 2. A 5-card hand is drawn from a deck of standard playing cards.

What's the probability that you get a heart ♥ ?

How many 5-card hands are there?
= C(52,5) = 2598960

How many 5-card hands have **at least one heart?**

Total # of deals – Number of deals with **no** hearts
= C(52,5) – C(52-13,5) = 2023203

So what's the requested probability?

2023203/2598960 ≈ 0.7785

Another approach

P[no heart] = P(card1 not a ♥ and card2 not a ♥ and … card5 not a ♥)
= P(card1 not a ♥) P(card2 not a ♥)…P(card5 not a ♥)
= $(3/4)^5$

P[get a heart] = 1 – P[didn't get a heart] = 1 – $(3/4)^5$ = 0.7627

What went wrong?

P[no heart] = (39/52)(38/51)(37/50)(36/49)(35/48) = P(39,5)/P(52,5)
≈ 0.7785

Interesting: P(39,5) / P(52,5) = (C(52,5) – C(39,5)) / C(52,5)

Problem 3. What's the chance that a 5-card deal will contain **at least two cards** with the same rank (a pair)?

Total # Of Deals – # of Deal where All 5 cards have different ranks
= C(52,5) – C(13,5) 4^5 = 1281072

So we want 1281072 / P(52,5) = 1281072 / 2598960 ≈ 0.4929

Problem 4. Describe the sample space and probability measure for dealing a poker hand?

Sample space has $|S| = C(52,5)$
We can regard the points of S as 5-elements subsets of
Cards = {2,3,4,5,6,7,8,9,J,Q,K,A} x {c,d,h,s}
Probability measure is uniform: $p(a) = 1/|S|$.

Fair coin

Problem 5. You flip a fair coin 100 times. What is the probability space and probability measure?

$S = \{0,1\}^{100}$
$p(a) = 2^{-100}$ for all $a \in S$.

Problem 6. You flip a coin 100 times. What is the probability of getting exactly 50 out of the 100 coin flips land heads? 51 out of the 100 coins?

$P(50\text{Heads}) = C(100,50) \; / \; 2^{100} \approx 0.07959$

$P(51\text{Heads}) = C(100,51) \; / \; 2^{100} \approx 0.07803$

Problem 7. You flip a coin 100 times. The (unfair) coin lands heads a fraction $p = 0.51$ of the time. Now what's the chance of getting 50 heads? 51 heads?

Now, what if the coin is biased?
Say that the coin lands **heads** with probability b **= .**51 and **tails** with probability $1 - b = .49$.
each flip independent of the rest.

$S = \{0,1\}^{100}$ (same as before, but now)

$p(x) = b^{\#1(x)} (1-b)^{\#0(x)}$

where $\#1(x)$ = number of 1-bits in the string x and
$\#0(x)$ = the number of 0-bits in the string x.

What's the probability of 50 and 51 heads now?

$P(\text{50Heads}) = C(100,50)\,(.51^{50})(.49^{50}) \approx 0.07801$
$\text{P}(\text{51Heads}) = C(100,51)\,(.51^{51})(0.49^{49}) \approx 0.07906$

Makes sense -- 51 heads should now be the most likely number, and things should fall off from there. Before, 50 heads was the most likely outcome.

Expectation

Def: A **random variable** is a function $X\colon S \to \mathbb{R}$ from the sample space to the reals. // Sometimes we allow a different codomain.
// Sometimes we use a special font for RVs, like X

Def: $E[X] = \sum_{a \in S} X(a)\, p(a)$ // **expected value** of X ("average value")
// Alternatively write $E(X)$ or $\mathbf{E}[X]$

Problem 8. Alice rolls a die. What do you expect the square of her roll to be?

Could be 1 .... could be a 36.

Definition: $E[X] = \sum_{a} X(a)\, p(a)$

So, in this problem,
$E[X] = 1(1/6) + 2^2(1/6) + 3^2(1/6) + \ldots + 6^2(1/6)$
$= 1/6(1+4+9+15+25+36)$
$= 91/6$
≈ 15.2

Problem 9. A multiple choice test with 5 answers per question takes off ¼ point for each wrong answer, gives 1 points for each right question. Maple has no idea what the answer is on question 5 so picks a random answer. The test has the right answer uniformly at random in any of the five locations. What is Maple's expected score on question 5?

First, let's recast it. Instead of maple picking a random answer, let's imagine maple picks a fixed answer, say 'a', and that the exam is designed to place the right answer at a random location. It amounts to the same.

Let X = Maple's score on problem 5. Sample space S = {a, b, c, d, e}. Each with $p(\text{a})=\ldots=p(\text{e})=1/5$. The RV X: {a, b, c, d, e}→{0,1} where $X(\text{a})=1$ and $X(\text{b})= X(\text{c})= X(\text{d})= X(\text{e})=0$

$$
\begin{aligned}
E[X] &= (1/5)\,1 + (1/5)\,(-1/4) + (1/5)\,(-1/4) + (1/5)\,(-1/4) + (1/5)\,(-1/4) \\
&= 1/5 + (4/5)(-1/4) \\
&= 1/5 - 1/5 \\
&= 0
\end{aligned}
$$

Problem 10. You roll a pair of fair dice. What's the chance of rolling an 8 if I tell you that both numbers came out even?

Method 1: Imagine the new probability space:

(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)
*** *** ***

So probability is 3/9 = 1/3

Method 2: A little more mechanically

A = "rolled an 8"
B = "both die are even"

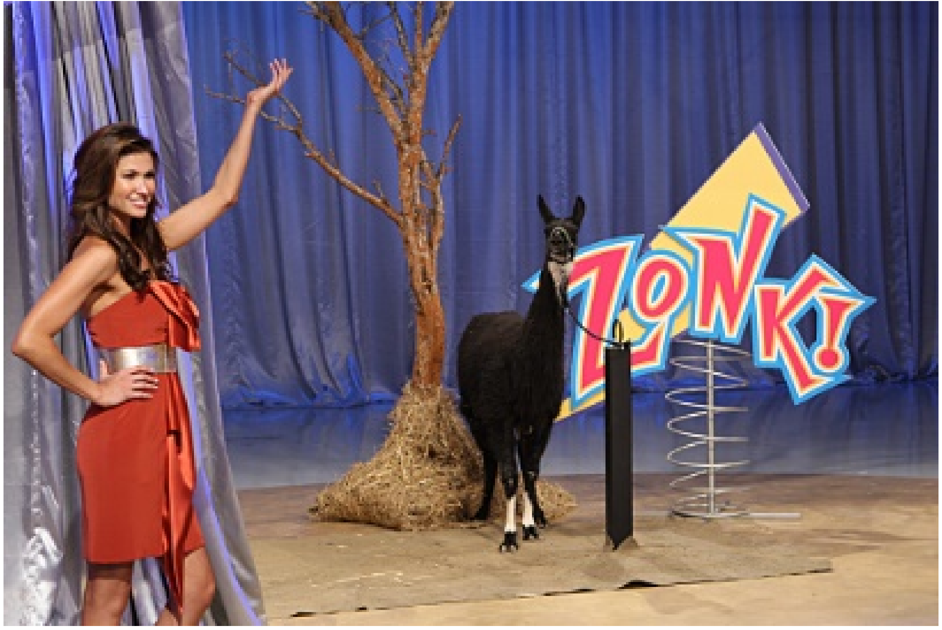
Def [conditional probability]:

$P(A \mid B) = P(A \cap B) / P(B)$ assuming $B \neq \emptyset$

$$
\begin{aligned}
P(A \mid B) &= P(A \cap B) / P(B) \\
&= (3/36) \;/\; (9/36) = 1/3
\end{aligned}
$$

Problem 11: Monty Hall Problem

Let's make a Deal (1963-1968)



```
 ========    ========    ========
 |      |    |      |    |      |
 | bad  |    | bad  |    | good |
 |      |    |      |    |      |
 ========    ========    ========
    1           2           3
```

A good prize is hidden behind a random curtain/door (junk behind the other two). You choose a door. The host opens one of the unselected doors that does NOT contain the good prize. Should you **switch** to the other door?

loc of good prize   which unselected door to open if host must choose
$S =$ {1, 2, 3} × {0, 1}   not really relevant

WIN = event that the contestant gets the good prize

Strategy STICK: choose door 1 and stick with it: P (WIN)=1/3

Strategy SWITCH: choose door 1 and then switch (always) when offered a chance.

Calculation by method 1:

(1,0) (2,0) (3,0) Second bit doesn’t matter.
Lose Win Win
(1,1) (2,1) (3,1)
Lose Win Win

P(Win) = 4/6 = 2/3

Calculation by method 2:

P(Win) = *P*(Win | initialCorrect) *P*(initialCorrect)
+ *P*(Win | initialIncorrect) *P* (initialIncorrect)

= 0 + 1 (2/3) = 2/3

Problem 12: Birthday phenomenon

Conditional probability uses: $P(A \mid B) = P(A \cap B) / P(B)$ assuming $B \neq \emptyset$

Proposition: $P(A) = P(A \mid B)\, P(B) + P(A \mid B^c)\, P(B^c)$

$$\frac{P(A \cap B)}{P(B)} \; P(B) \; + \; \frac{P(A \cap B^c)}{P(B^c)} \; P(B^c)$$

n=23 people gather in a room.
What's the chance that some two have the same birthday?
Assume nobody born 2/29, all other birthdays equiprobable.
$S = [1..365]^{23}$ **Named events:**
D_1 = entire sample space
D_2 = people 1,2 have distinct birthdays
D_3 = people 1,2,3 have distinct birthdays
…
D_{23} = people 1, 2, …, 23 have distinct birthdays
C_{23} = some two people among 1, 2, …, 23 have the same birthdays

$P\,[D_{23}] = P[D_{23} \mid D_{22}]\, P[D_{22}]$ ~~$+ P[D_{23} \mid !D_{22}]\, P[!D_{22}]$~~
$= P[D_{23} \mid D_{22}]\, P[D_{22} \mid D_{21}]$
$= P[D_{23} \mid D_{22}]\, P[D_{22} \mid D_{21}]\, P[D_{21} \mid D_{20}]$
…
$= P[D_{23} \mid D_{22}]\, P[D_{22} \mid D_{21}] \ldots P[D_2 \mid D_1]$

$= (1 - 22/365)(1\text{-}21/365) \ldots 1(1\text{-}1/365)\ 1$

$$= \frac{343}{365}\ \frac{344}{365}\ \frac{345}{365} \ldots \frac{365}{365}$$

$= (1/365)^{23}\ (365\ 364 \ldots 343)$
≈ 0.493

$P(C_{23}) = 1 - P(D_{23})$
$= 1 - (1\text{-}1/365)(1\text{-}2/365) \ldots (1\text{-}22/365)$
$\approx 1 - 0.493$
$= 0.507$