Problem Set 2 – Due Wednesday, January 19, at 5 pm

1. (a) Compute $155 + 71$ (both numbers written in decimal) by converting both numbers into binary, adding them up as binary numbers (mimic what you do in decimal), and then converting the sum back to decimal. (b) When we talked about representing numbers in other bases, we were always talking about representing natural numbers. Think about adding a decimal point, so that you can represent floating point values in binary. What ought to be the binary representation of 20.625? What about 20.2?

2. John is thinking of a 100-digit decimal number. How many bits will it be if he converts it to binary? Be as exact as you can.

3. This problem assumes you know how to play "tic-tac-toe". If you don't, ask someone.

Suppose you represent the current state of a tic-tac-toe board $b = b_1 b_2 \cdots b_9$ with nine characters, each $b_i \in \{\texttt{X}, \texttt{O}, \texttt{-}\}$, with a naming convention of

$$\begin{matrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{matrix}$$

Write a boolean formula for each of the following five conditions: player-`X` has won; player-`O` has won; someone has won; the game is a draw; the game is over. You may assume that the board is "valid"—that it is a position that could actually arise in a game. You write things like $b_3 = \texttt{X}$ and you can use any logical connectives you like, but no quantifiers.

4. Recall that a binary predicate $P\colon \mathbb{B}^2 \to \mathbb{B}$ is *functionally complete* if, for any $n \geq 1$ and any function $F\colon \mathbb{B}^n \to \mathbb{B}$, it is possible to realize F using nothing but P-gates.

"I think your notion is too strict," Constance objects. "If you were trying to build a digital circuit using P-gates, wouldn't you also have available reference values of 0 and 1, which are just the ground (0V) and a reference voltage on the chip? In defining functional completeness, you should account for that."

Using Constance's idea, let's say that a binary predicate $P\colon \mathbb{B}^2 \to \mathbb{B}$ is *almost functionally complete* if, for any $n \geq 1$ and any function $F\colon \mathbb{B}^n \to \mathbb{B}$, it is possible to realize F using nothing but P-gates, the constant 0, and the constant 1.

Find a boolean function $P\colon \mathbb{B}^2 \to \mathbb{B}$ that is almost functionally complete but not functionally complete. Explain why it is so.

5. Our heroine arrives at a two-doored gateway. Behind one door lies a magical castle. Behind the other, certain death. One door is protected by a guard that always lies; the other, by a guard that always tells the truth. Ask a single question to one of the guards that will reveal which door leads to the castle.

6. A sadistic guard neatly lines up his 100 prisoners in height order, prisoner P_0 the tallest, prisoner P_{99} the shortest, with each prisoner P_i facing the back of prisoner P_{i+1}. (The guard, having majored in CS, prefers to index his prisoners starting at 0.) So situated, prisoner P_i can see the head of each prisoner P_j with $j > i$. The guard shouts: "I shall now place hats atop your stupid little heads. Either a black hat or a white hat. Then I shall ask you, starting at prisoner P_0 and moving my way to prisoner P_{99}, if your *own* hat is black or is white. Shout it out. If you answer correctly, you will live another day. If you answer incorrectly, I will shoot you without delay. Bang! If you try to communicate with one another once our game begins, I will shoot you all."

The prisoners briefly confer and agree to a strategy to minimize their losses. (As luck would have it, one prisoner had taken ECS20 and knew exactly what to do.) Describe it. *Hint: Think parity/XOR.*