Problem Set 5 – Due Wednesday, February 9, at 5pm

1. Figure out which of the following relations $\sim$ are equivalence relations. As always, explain your answers.

   (a) $x \sim y$ if x and y are people who were born on the same day of the week.

   (b) $x \sim y$ if x and y are nonempty strings that contain a common character.

   (c) $x \sim y$ if x and y are points in the plane that are equidistant to the origin.

   (d) $L \sim L'$ if L and L' are parallel lines in the plane. Assume that L and L' are parallel means that they have the same (possibly infinite) slope.

   (e) $L \sim L'$ if L and L' are parallel lines in the plane. Assume that L and L' are parallel means that they do not intersect.

2. For $a, b \in \mathbb{R}$ define $a \sim b$ if $a - b$ is an integer.

   (a) Prove that $\sim$ is an equivalence relation.

   (b) What is the equivalence class (block) containing 3? What is the equivalence class of 3.14? That is, clearly describe the sets $[3] = \{y : 3 \sim y\}$ and $[3.14] = \{y : 3.14 \sim y\}$.

   (c) If you wanted to select for each equivalence class of $\sim$ a canonical representative, or *name*, what name would you use? For example, what would be the canonical name for $[\pi]$?

3. Functions don't have to have familiar domains. Define at least three "interesting" functions whose domain $B \subseteq \{\mathsf{X}, \mathsf{O}, \text{-}\}^9$ is the set of all valid tic-tac-toe board positions, whether the game is incomplete or finished, that can arise in a properly-played game. Assume X moves first. Clearly specify each function's domain and target using standard notation. Have varying targets for your different functions.

   Your functions should capture something significant about the game. While "interesting" is obviously subjective, make your functions capture something nontrivial about the input $b \in B$.

   For each function f that you describe, specify the meaning of $f(s)$ and $f(u)$ where $s =$ `---------` is the initial board and $u =$ X-OXO-XOX is a completed game. You do not have to calculate function values that are not obvious.

4. We can denote a permutation on $[0..9]$ as with
   $$\sigma = \langle 5\ 2\ 7\ 8\ 3\ 0\ 1\ 9\ 6\ 4 \rangle = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 2 & 7 & 8 & 3 & 0 & 1 & 9 & 6 & 4 \end{pmatrix}$$
   for the function that sends 0 to 5, sends 1 to 2, and so on. Similarly, let
   $$\pi = \langle 3\ 9\ 4\ 5\ 0\ 2\ 7\ 8\ 6\ 1 \rangle = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 9 & 4 & 5 & 0 & 2 & 7 & 8 & 6 & 1 \end{pmatrix}.$$

   (a) Suppose we multiply permutations by composing them right-to-left, so $\pi\sigma = \pi \circ \sigma$ means to do σ first, then π. Compute both $\pi \circ \sigma$ and $\sigma \circ \pi$. Are they the same?

   (b) Lets write σ^n for multiplying σ by itself $n - 1$ times. Let $\sigma^1 = \sigma$ and let σ^0 be the identity permutation. Explain what happens to 1 as you apply σ repeatedly. What is $\sigma^{100}(1)$?

   (c) Invent a nice alternative way to represent (that is, write) a permutation on $[0..n-1]$, a way that more conveniently shows what happens when you iterate. Express σ and π with your alternative notation.

   (d) Compute σ^{100}. Write your answer your new notation and in our original notation.

(e) Will σ^n ever equal the identity permutation e (where $e(n) = n$)? If so, for what smallest value of n? Same question for π.

(f) Find a permutation σ^{-1} you can multiply σ by get the identity permutation e. Find a permutation π^{-1} you can multiply π by get the identity permutation e. When you multiply by the inverse, does it matter if you multiple on the left or on the right?

5. *Let $N = (38\,007\,000\,041)_9$ (that is, base-9). What is $N \bmod 5$? Don't use a calculator or online tool for this; the problem has a simple pencil-and-paper solution using only things you know.