Problem Set 6 – Due Wednesday, February 23, at 5pm

1. For $a, b \in \mathbb{R}$ with $a < b$, let $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$. Given a collection of intervals $\mathcal{L} = \{[a_i, b_i] : i \in \mathbb{N}\}$, their *total length* is $\ell = \sum_i (b_i - a_i)$. The intervals of $\mathcal{L}$ are said to **cover** a set $A \subseteq \mathbb{R}$ if $A \subseteq \bigcup_i [a_i, b_i]$.

(a) Find a collection of intervals $\mathcal{L}$ whose total length is at most 1 but which covers all of $\mathbb{Q}$, the rational numbers. *Hint:* $1/2 + 1/4 + 1/8 + \cdots = 1$.

(b) Modify your collection of intervals $\mathcal{L}$ so that their total length is now at most 0.1, say, but they still cover $\mathbb{Q}$. Generalize further. *Hint:* $p + p^2 + p^3 + \cdots = \frac{p}{1-p}$ *when* $|p| < 1$.

2. Considered a lucky number, the Thai government has issued coins of 9 baht. Show that, for all sufficiently large numbers N (just how large?), you can dispense N baht using only 9 baht and 10 baht coins.

3. Prove that $3 \mid n^3 + 2n$ for every $n \ge 1$. Then give a second, different proof.

4. Prove that $2^n > 10n^2$ for all sufficiently large integers n.

***5.** Suppose you create a collection of points in the plane, S, by asserting that $(0, 0) \in S$ and that whenever $(x, y) \in S$ then so are: $(-x, -y)$, (y, x), $(2y, 2x)$, $(x + 5.5, y + 3.5)$, $(3x - y + 7, 2y - 2x - 1)$, and $(x + 2y, 2x + y)$. Nothing else is in S. Prove that $(1234, 4321) \notin S$. *Hint:* An invariant.

6. How many (positive) divisors $d(n)$ does the number $n = 2\,450\,250\,000$ have? Don't use any online tools that for computing $d(n)$.

***7.** A thousand Christmas lights, indexed [1..1000], decorate the yard of Natasha's home. All the lights are initially OFF. A mischievous ferret goes to each light $n \in [1..1000]$ and toggles the switch. Now the 1000 lights are ON. It looks pretty. Yet thinking the better of it, the ferret decides to go back and toggle the switch for each even-indexed light. Now there will be 500 lights ON. Next our friend toggles each light whose index is a multiple of 3. After that, it toggles each light whose index is a multiple of 4. And so on. On round 1000, it toggles the switch only for light number 1000. After that, the exhausted ferret goes to sleep. How many lights illuminate the sleeping creature? Fully explain your answer.


The ferret, Mustelo, that lives in Problem 7.