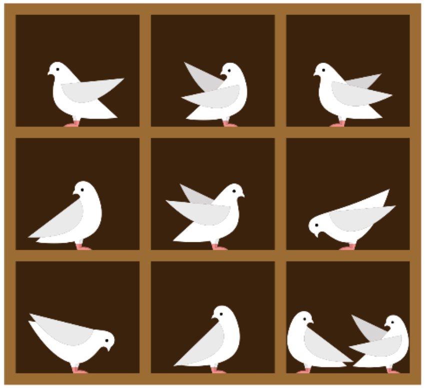


Lecture 1T

Today:

- ▢ Course basics
- ▢ Example problems
    - Counting paths
    - Five riffle shuffles won't well-mix a 52-card deck

Phillip Rogaway

ECS20.A webpage:
https://web.cs.ucdavis.edu/~rogaway/classes/20/winter22/

Administrative stuff

- Turn on your camera, please!

- Read the syllabus!
- Get setup on Gradescope and Piazza. Use your campus ID for this.
- Get setup on Overleaf (and/or install LaTeX local) (highly recommended)
- Get setup on zyBooks (highly recommended)
- First problem set is due next Wednesday

- TAs: **John Chan** and **Zane Rubaii**
- Graders: **Teo Anderson** and **Kev Rockwell**

When we resume meeting in person:

- No phones! In your bag. Laptops, too.
- Proper mask, properly worn. N95 requested.

"Discrete Math" isn't really a "thing"

Logic

Information theory

Set theory

Cryptography

Number theory

Game theory

Theory of Computation

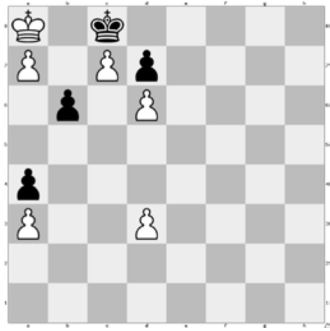
Combinatorics

Probability
(on finite probability spaces)

Complexity theory

Graph theory

Operations research



Perhaps there is an implicit viewpoint undergirding the kind of math that gets lumped as "discrete math".

A belief that we gain by conceptualizing the world as having separate and distinct possibilities.

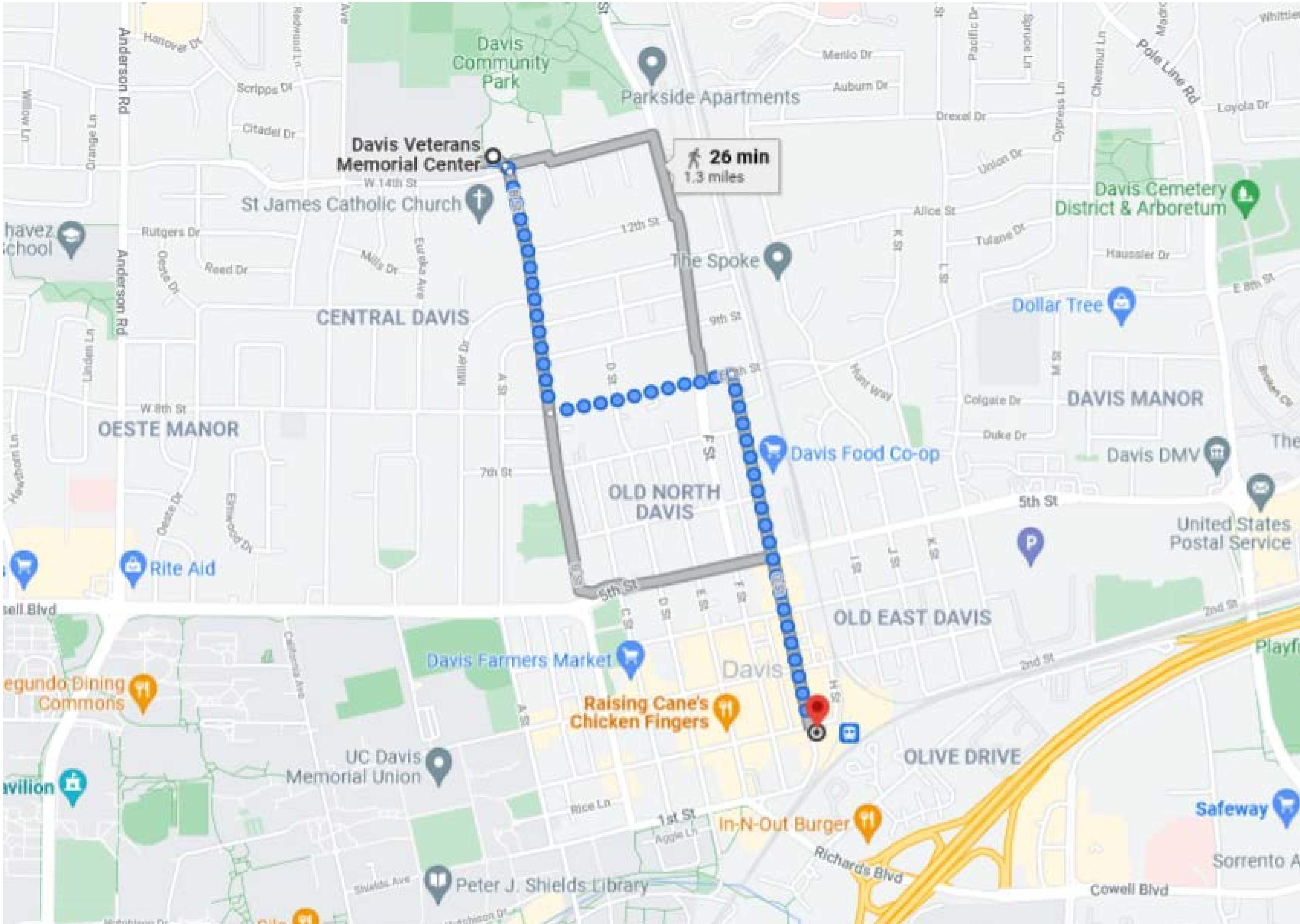
$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$

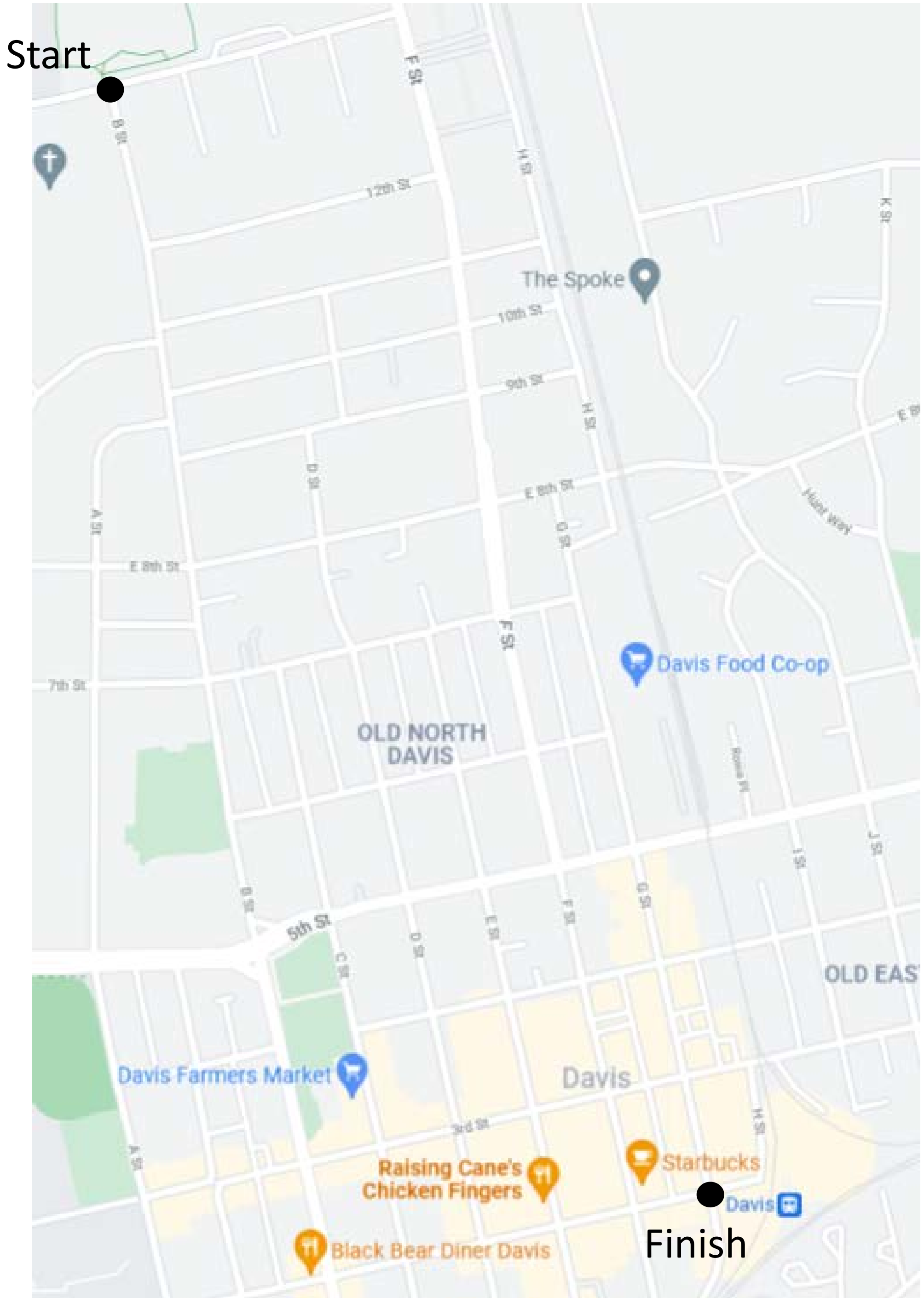
A hierarchy, learned as as truth slowly revealed

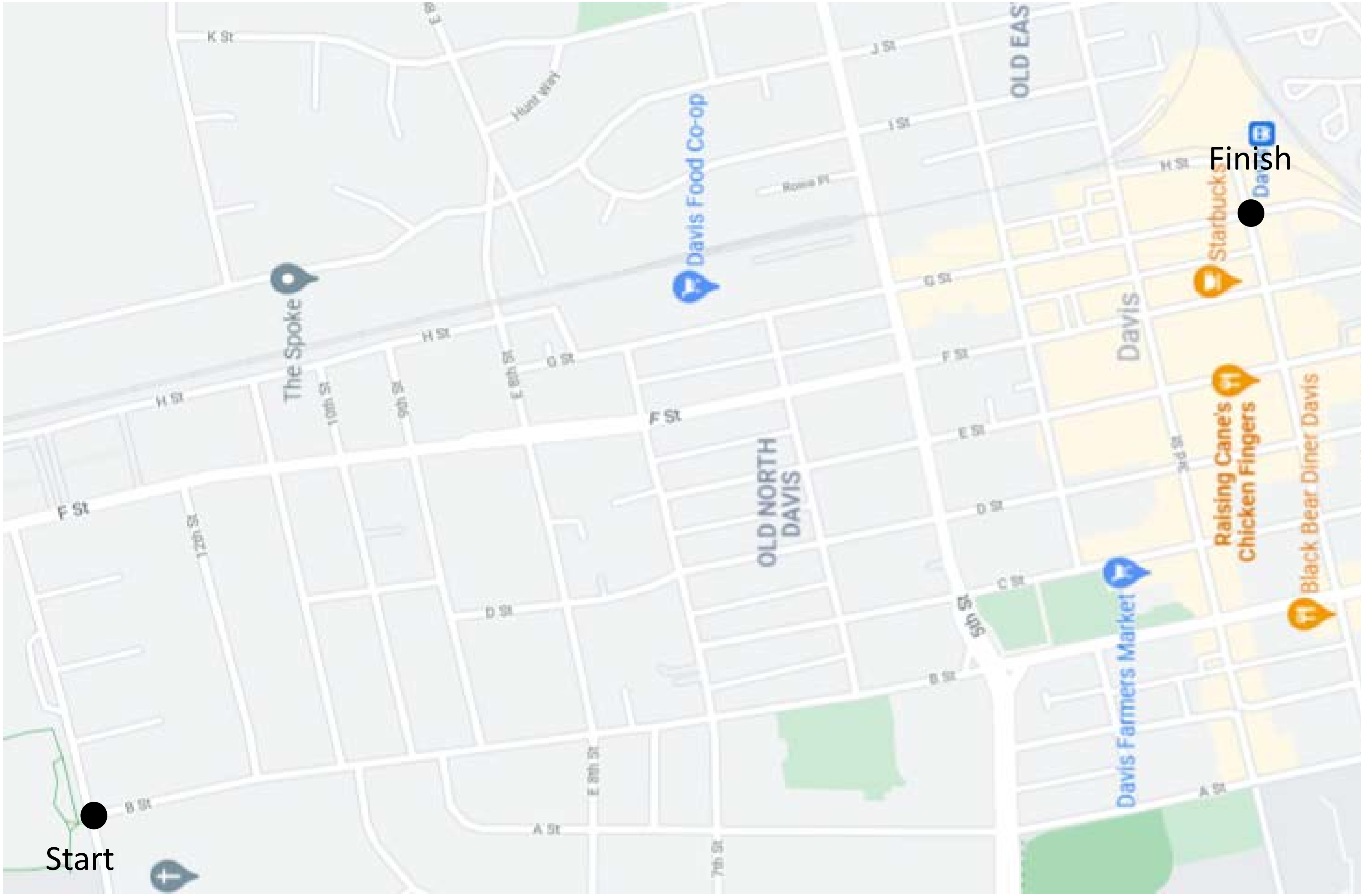
$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{R}$

Three different places to live and work

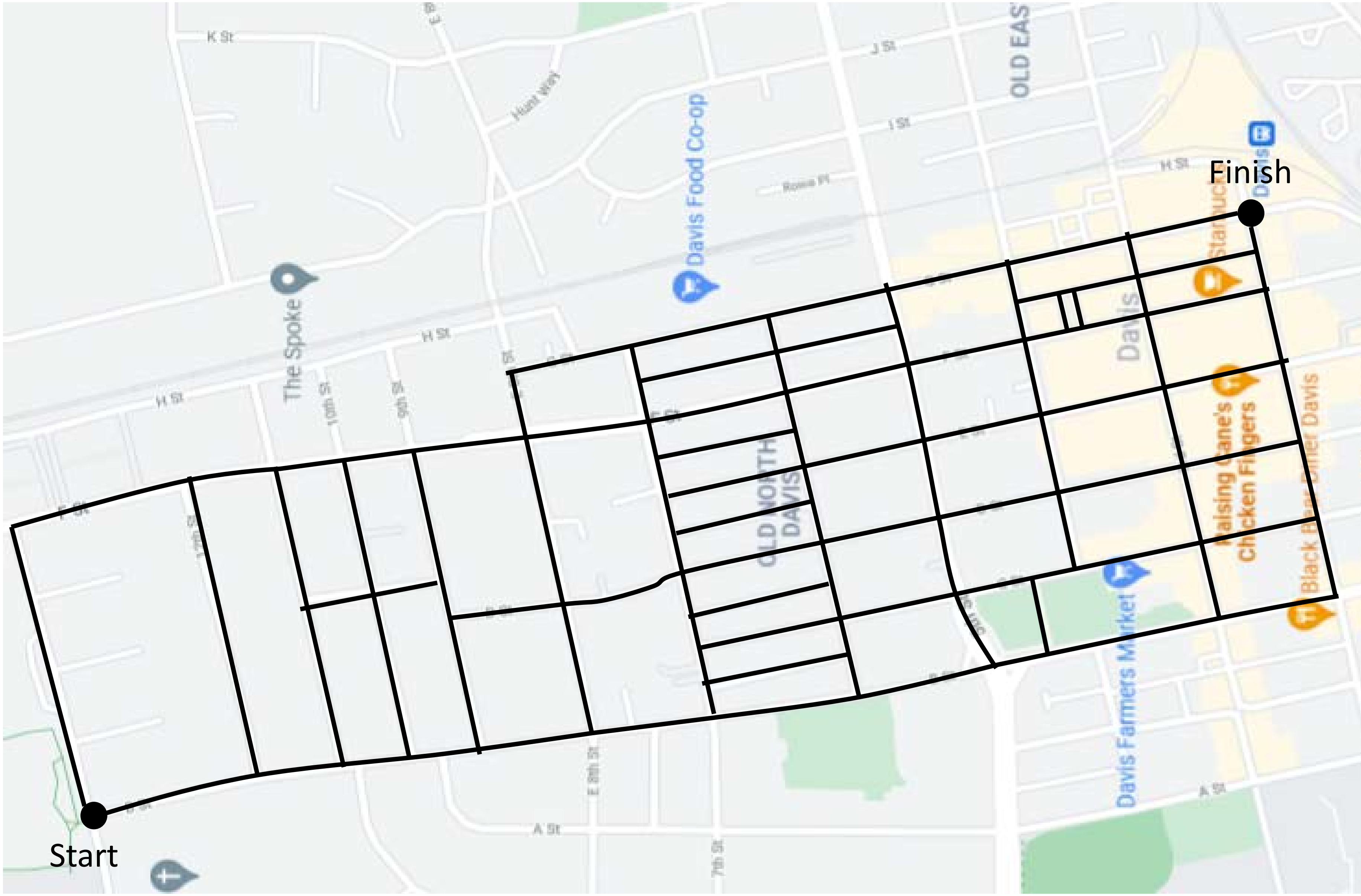
Example 1 How many ways are there to walk **from** VMC (14^{th} & B) **to** Pachamama Coffee Shop (C & 3^{rd}) ?

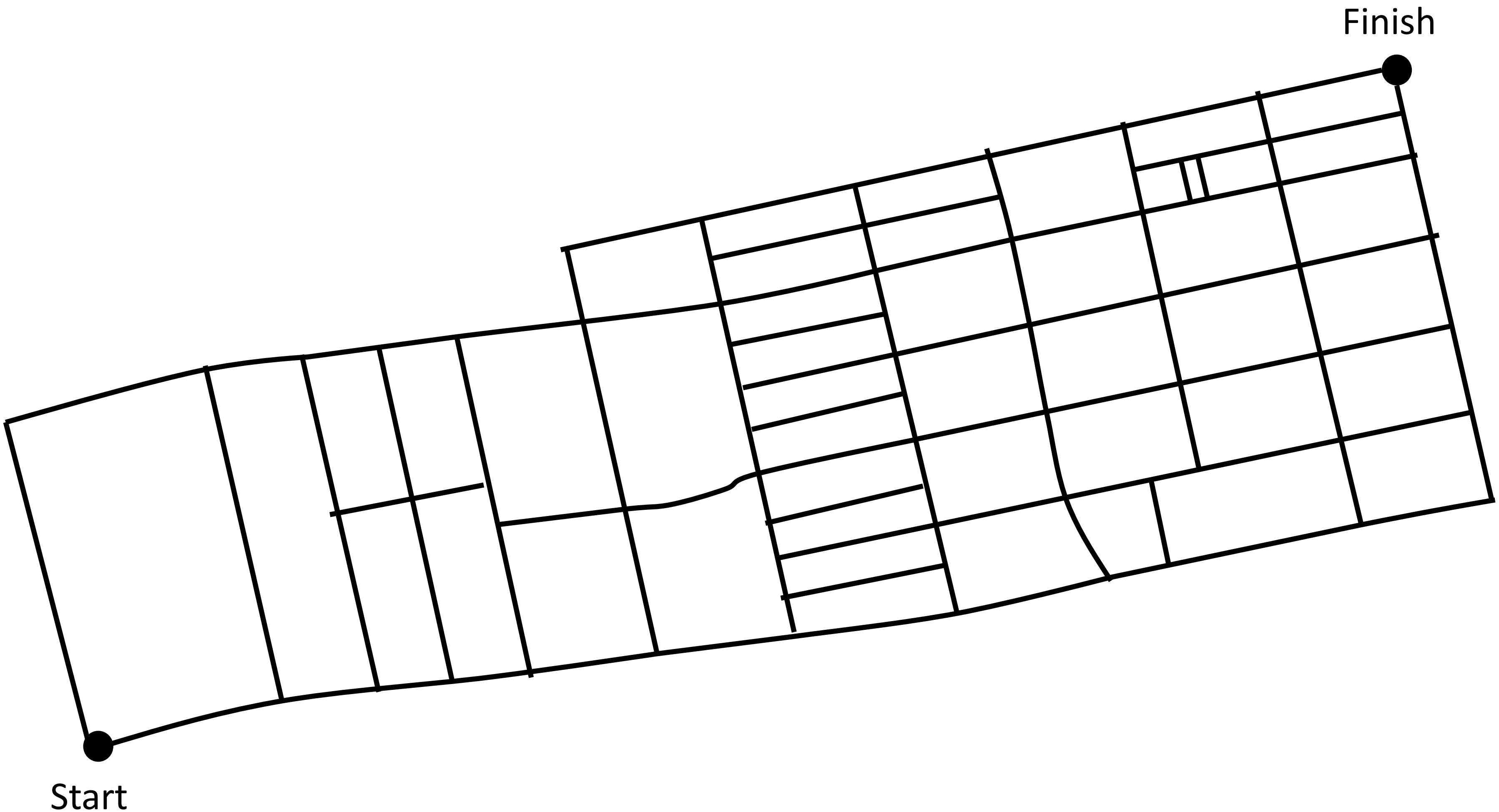



Start
Finish
12th St
10th St
9th St
E 8th St
7th St
5th St
3rd St
A St
B St
C St
D St
E St
F St
G St
H St
I St
J St
K St
Hunt Way
Rose Pl
The Spoke
Davis Food Co-op
OLD NORTH DAVIS
OLD EAS
Davis
Davis Farmers Market
Raising Cane's Chicken Fingers
Starbucks
Black Bear Diner Davis

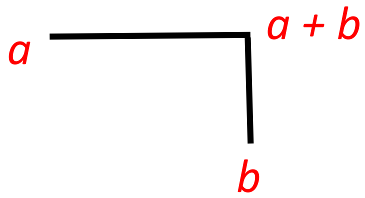


Finish
Start
K St
J St
I St
H St
Rowe Pl
G St
F St
E St
D St
C St
B St
A St
12th St
10th St
9th St
E 8th St
7th St
5th St
3rd St
Hunt Way
OLD NORTH DAVIS
OLD EAST
Davis
The Spoke
Davis Food Co-op
Starbucks
Raising Cane's Chicken Fingers
Black Bear Diner Davis
Davis Farmers Market

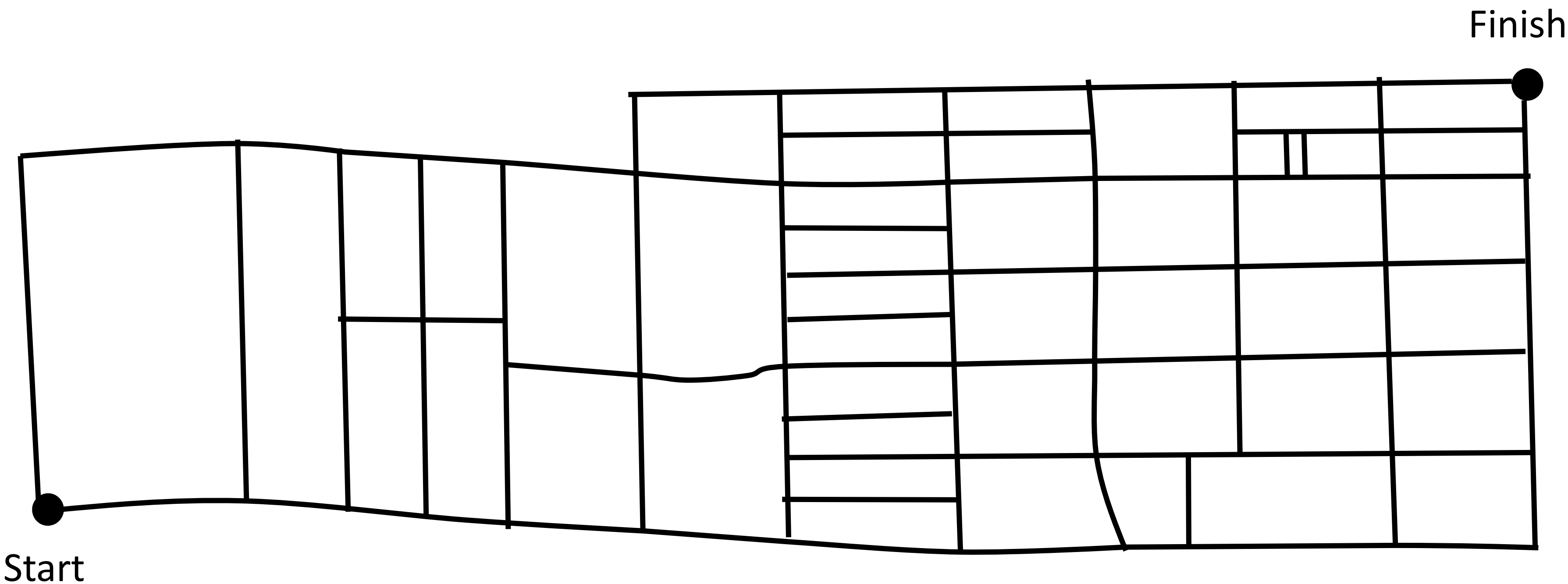

Start
Finish
K St
J St
I St
H St
Rosa Pl
The Spoke
Davis Food Co-op
OLD EAST
OLD NORTH DAVIS
Davis
Davis Farmers Market
Raising Cane's Chicken Fingers
Black Bear Diner Davis
A St
10th St
9th St
E 8th St
7th St

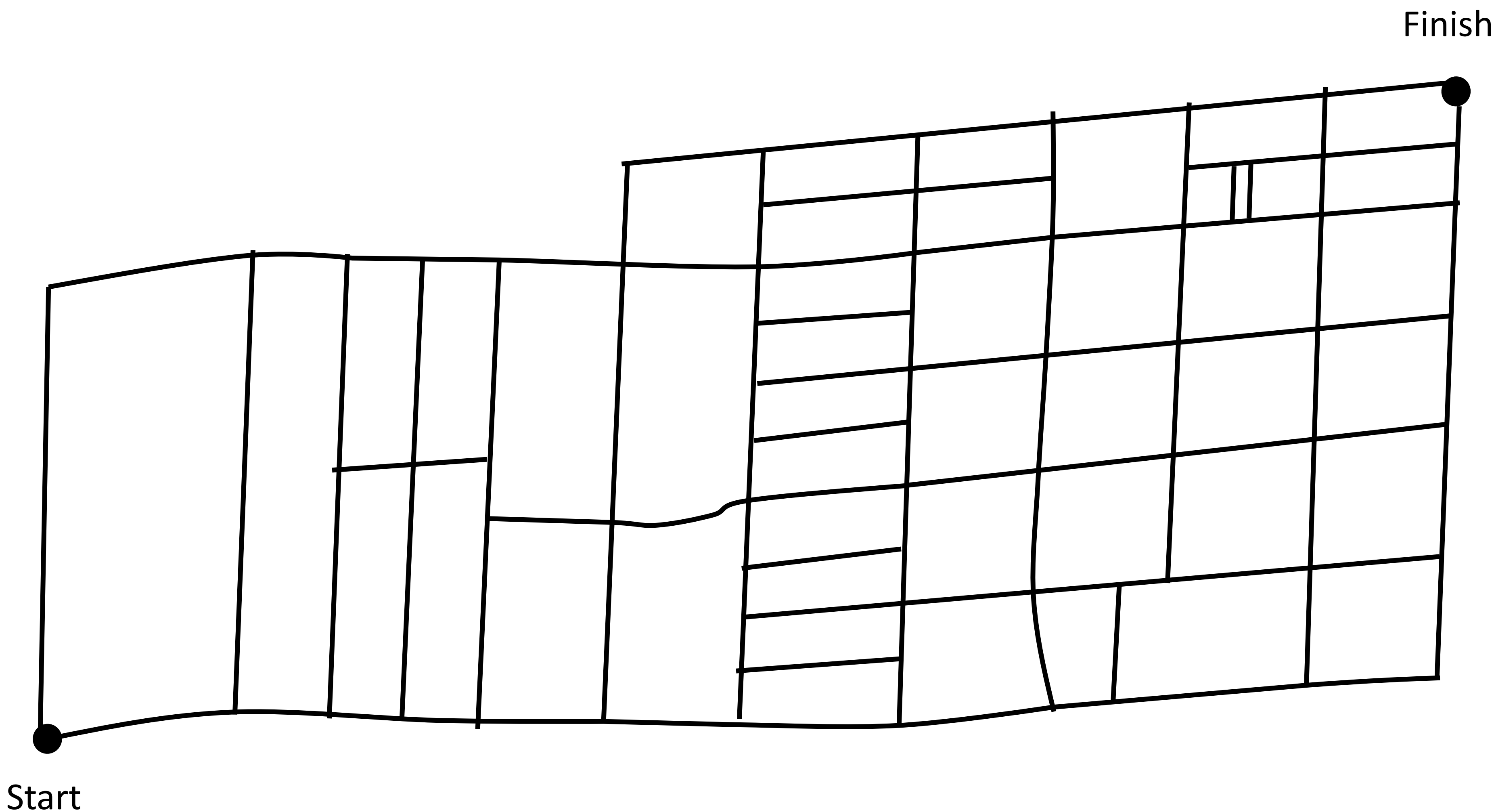

Finish
Start

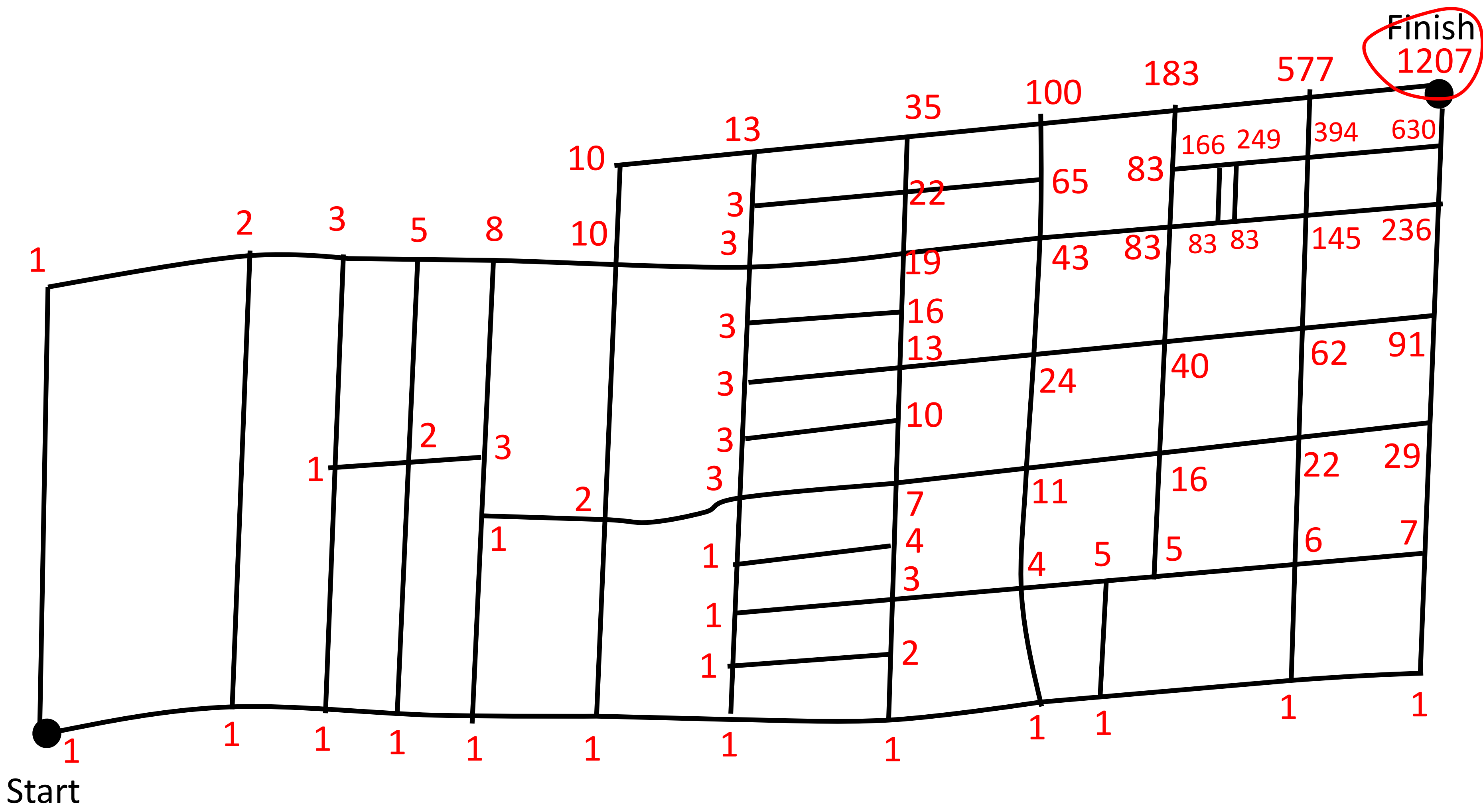
The number of paths to a point is the **sum** of the number of paths to its closer neighbors.



The number of paths to the Start point is 1.



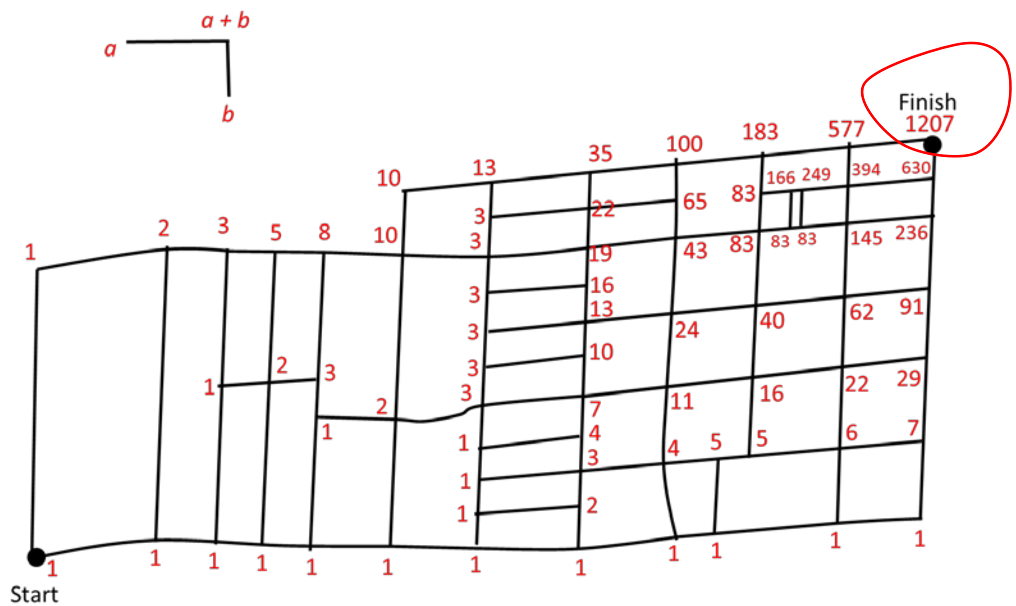

Finish
Start


Finish
1207
Start

How many ways are there to walk

from VMC (14th & B) **to** Pachamama Coffee Shop (C & 3rd) ?

Mind the Gap!

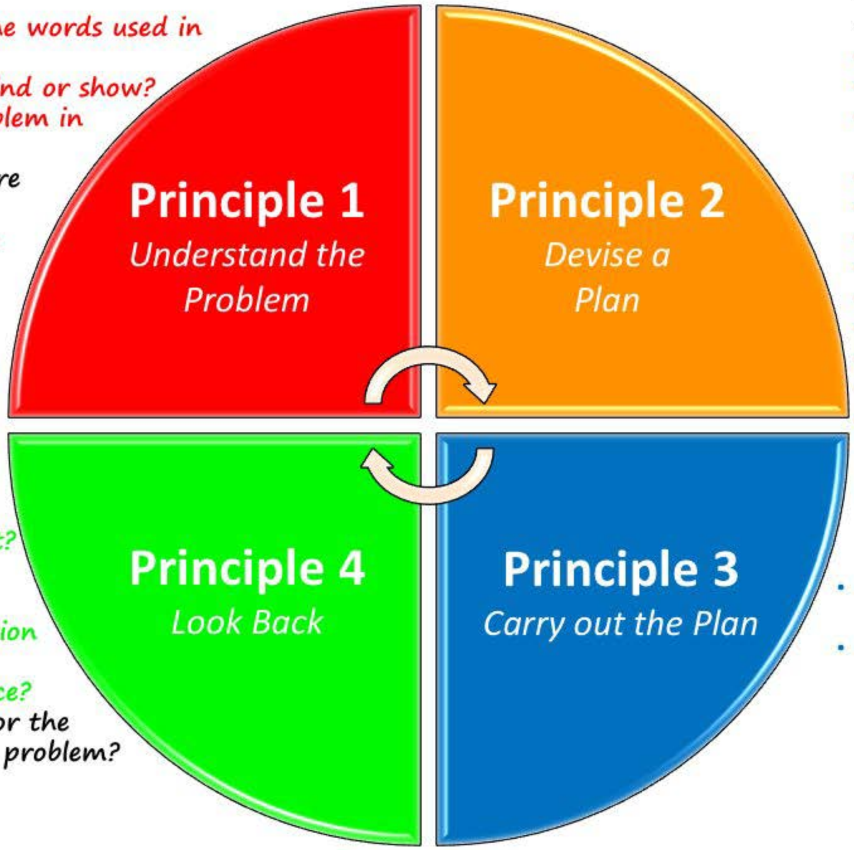


Look back and
ask follow-up questions.

George Polya, *How to Solve It* (1945)

- Do you understand all the words used in stating the problem?
- What are you asked to find or show?
- Can you restate the problem in your own words?
- Can you think of a picture or diagram that might help you understand the problem?
- Is there enough information to enable you to find a solution?

- Draw a picture
- Use a model
- Use direct reasoning
- Solve an equation
- Solve a simpler problem
- Guess and check
- Make an orderly list
- Eliminate possibilities
- Use symmetry
- Consider special cases
- Look for a pattern
- Work backwards
- Use a formula
- Be ingenious



- Persist with the plan that you have chosen.
- If it continues not to work discard it and choose another.

- Can you check the result?
- Can you check the argument?
- Can you derive the solution differently?
- Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

Based on George Polya 'How To Solve It' (1945)
https://math.berkeley.edu/~gmelvin/polya.pdf

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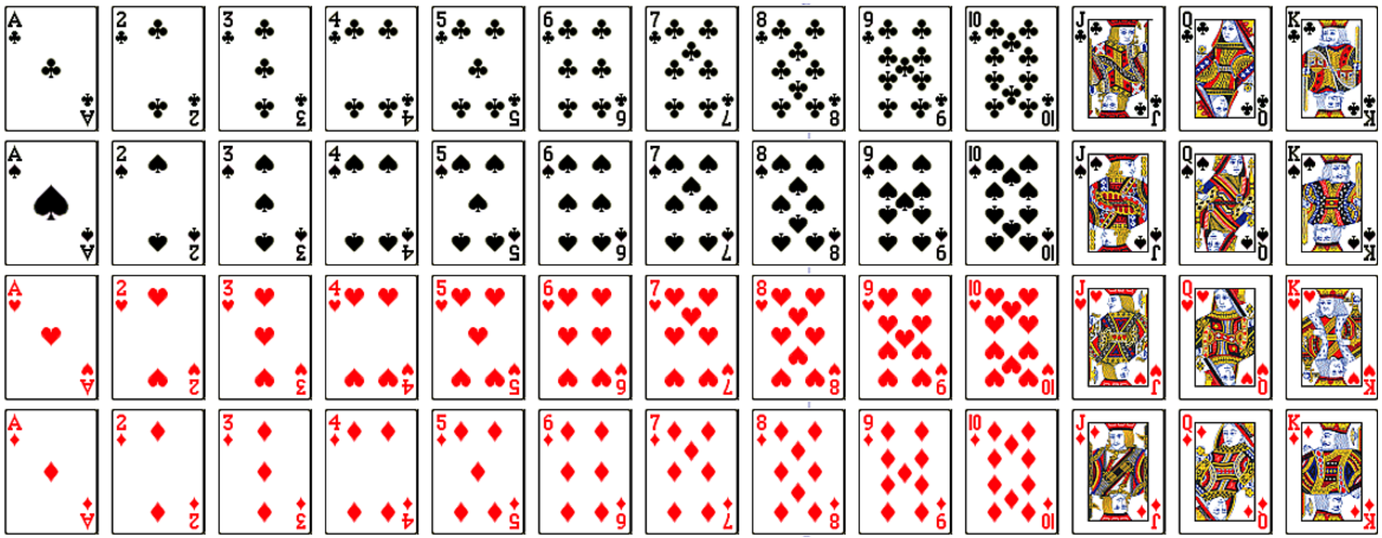
MaxTheMaths
Learners • Teachers • Leaders

Example 2 **Claim:** It takes at least six riffle shuffles to nicely mix a deck of cards. Five won't do.



(1 2 3 4 5 6 7 8 9 10 11 12 … 46 47 48 49 50 51 52)

More specific claim:

Five riffle shuffles can never transform the sequence of cards

$$(1 \quad 2 \quad 3 \quad \ldots \quad 50 \quad 51 \quad 52)$$

To the sequence of cards

$$(52 \quad 51 \quad 50 \quad \ldots \quad 3 \quad 2 \quad 1)$$

s **= (7 3 8 4 9 1 2 5 6)**

Definition: Let *s* be a sequence (=list) of integers.
A **rising sequence** in *s* is a
<u>maximal</u> <u>subsequence</u> of *s* of <u>consecutive numbers</u>

(8 9 2 5)

(4 5 6)

(3 4 5 6)

Claims:

- The number of rising sequences in any sequence *s* is well defined.
- If you riffle shuffle a sequence *s*, the number of rising sequences in it will **at most double**.
- There is **1** rising sequence in 1,2, 3, …, 51, 52 is 1.
- There are **52** rising sequences in 52, 51, …, 3, 2, 1.
- So maximal number of rising sequences
  after 1 riffle shuffle is 2
  after 2 riffle shuffles is 4
  after 3 riffle shuffles is 8
  after 4 riffle shuffles is 16
  after 5 riffle shuffles is 32
  after 6 riffle shuffles is 52

In particular, you **can't get** 52, 51, …, 2, 1 after 5 shuffles starting from 1, 2, …, 52.
The deck can't be adequately mixed.

The Annals of Applied Probability
1992, Vol. 2, No. 2, 294–313

TRAILING THE DOVETAIL SHUFFLE TO ITS LAIR

By Dave Bayer[1] and Persi Diaconis[2]

Columbia University and Harvard University

We analyze the most commonly used method for shuffling cards. The main result is a simple expression for the chance of any arrangement after any number of shuffles. This is used to give sharp bounds on the approach to randomness: $\frac{3}{2}\log_2 n + \theta$ shuffles are necessary and sufficient to mix up n cards.

Key ingredients are the analysis of a card trick and the determination of the idempotents of a natural commutative subalgebra in the symmetric group algebra.

1. Introduction. The dovetail, or riffle shuffle is the most commonly used method of shuffling cards. Roughly, a deck of cards is cut about in half and then the two halves are riffled together. Figure 1 gives an example of a riffle shuffle for a deck of 13 cards.

A mathematically precise model of shuffling was introduced by Gilbert and Shannon [see Gilbert (1955)] and independently by Reeds (1981). A deck of n cards is cut into two portions according to a binomial distribution; thus, the chance that k cards are cut off is $\binom{n}{k}/2^n$ for $0 \le k \le n$. The two packets are then riffled together in such a way that cards drop from the left or right heaps with probability proportional to the number of cards in each heap. Thus, if there are A and B cards remaining in the left and right heaps, then the chance that the next card will drop from the left heap is $A/(A+B)$. Such shuffles are easily described backwards: Each card has an equal and independent chance of being pulled back into the left or right heap. An inverse riffle shuffle is illustrated in Figure 2.

Table 1
Total variation distance for m shuffles of 52 cards

m	1	2	3	4	5	6	7	8	9	10
$\|Q^m - U\|$	1.000	1.000	1.000	1.000	0.924	0.614	0.334	0.167	0.085	0.043

Problem-Solving Techniques

1. Reformulate to something equivalent
2. Generalize
3. Work out special cases. Small cases. Look for patterns.
4. Name things (e.g., introduce variables)
5. Create tailor-made definitions
6. Draw pictures
7. Think recursively
8. Adopt a playful attitude
9. Forget pattern-matching
10. But look for echoes
11. Know what you know (don’t fool yourself, don’t try to fool others)
12. Give serious attention to exposition. Never turn in a first draft.
    Critically read what you write.