Lecture 5 (3T)  
Proofs

Announcements  
- Quiz 1 performance wasn’t bad  
- Zane to review of logic next week in discussion

Today:  
☐ Proofs illustrating sundry techniques

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Problem-Solving Techniques

1. Reformulate to something equivalent
2. Generalize
3. Work out special cases. Small cases. Look for patterns.
4. Name things (e.g., introduce variables)
5. Create tailor-made definitions
6. Draw pictures
7. Think recursively
8. Adopt a playful attitude
9. Forget pattern-matching
10. But look for echoes
11. Know what you know (don’t fool yourself, don’t try to fool others)
   Critically read what you write.
2. Proofs

2.1 Mathematical definitions

2.2 Introduction to proofs

2.3 Best practices and common errors in proofs

2.4 Writing direct proofs

2.5 Proof by contrapositive

2.6 Proof by contradiction

2.7 Proof by cases

- Follow-your-nose proof
- Introduce-the-right-extra-thing proof
- Draw-the-right-picture proof
- Find-a-good-counterexample proof
- Find-a-good-representation proof
- Break-into-good-cases proof
- Reduce-to-known-result proof
Example 1: $\sqrt{2}$ is irrational.
Example 2: There are irrational numbers $a$ and $b$ such that $a^b$ is rational.
Example 3: In playing tic-tac-toe, if the first player moves to a corner then the second player must take the center, or else the first player can force a win.
Example 4: The area of a triangle with three acute angles is $bh/2$ where $b$ is the triangle’s base and $h$ is its height.
Example 5: \[ 1 + 2 + \cdots + n = \]
Example 6:  \[ 1 + 3 + 5 + \cdots + 2n - 1 = \]
Example 7:

20 random cards are placed in a row, all face down. A move consists of turning a face-down card face-up and turning over the card, if any, immediately to the right. Show that no matter what the choice of cards to turn, this sequence of moves must terminate.
Example 8: [MIT book, chapter 11]
Who, on average, has more opposite-gender partners: men or women? ...

In one of the largest [studies], researchers from the University of Chicago interviewed a random sample of 2500 people over several years ... Their study, published in 1994, ... found that men have on average 74% more opposite-gender partners than women.

Other studies have found that the disparity is even larger. In particular, ABC News claimed that the average man has 20 partners over his lifetime, and the average woman has 6, for a percentage disparity of 233%. The ABC News study [claimed] a 2.5% margin of error...

Proposition: In a bipartite graph \( G=(V_1, V_2, E) \) with \( m \) edges and \( n \) vertices on each side of the bipartition, the average degree of a vertex in \( V_1 \) is the same as the average degree of a vertex in \( V_2 \).
Example 9:
Show that any party with 6 people will contain a group of 3 mutual friends or a group of 3 mutual non-friends.
Example 10: The last theorem wouldn’t work for 5 people.

Namely: you **could** have a party with **five** people and where **no three** are mutual **friends** and **no three** are mutual **non-friends**.
Example 11: Show that \(\{\rightarrow, \oplus\}\) is functionally complete.
Some Truths about Proofs

1. Finding proofs is not mechanical; it is an art.
2. Mathematical discovery is more than proofs: guessing and discovering results.
   Only in a class setting do proofs come as “prepackaged” task.
3. Don’t get so obsessed with rigor that you fail to develop and refine intuition – and to err.
4. Proofs evolve. They can be quite dialectical.
5. Intuition can be lost in a refined, succinct proof.
   Proofs are not “born” in such a manner.
6. You can’t prove what doesn’t make sense to you.
   Don’t even try to prove something until you get to the point of the language and claim making sense.