Problem Set 4 Solutions

Problem 1. *Give a context-free grammar for the language*

$$L = \{a^n b^m : n \neq 2m\}.$$

Is your grammar ambiguous?

A grammar for the language is

$$S \to aaSb \mid A \mid B \mid ab$$
$$A \to aA \mid a$$
$$B \to bB \mid b$$

This grammar is unambiguous; it is not hard to prove that every string in the langauge has one and only one parse tree.

Problem 2. *Give a context-free grammar for the language*

$$L = \{x \in \{0,1\}^* : \ x \text{ has the same number of 0's and 1's}\}$$

Is your grammar ambiguous?

A grammar for the language is

$$S \to S0S1S \mid S1S0S \mid \varepsilon$$

This grammar is ambiguous; for example, 0101 has two different parse trees (draw them!).

Problem 3. *Prove that the context-free languages are closed under reversal.*

Construct from the CFG G a new grammar G' by replacing every rule $A \to \alpha$ by $A \to \alpha^R$. It is easy to see that $L(G') = (L(G))^R$.

Problem 4. *Prove that* $L = \{a^i b^j c^k : j = \max\{i, k\}\}$ *is not context free.*

Suppose for contradiction that L were context free. Let N be the "N" of the pumping lemma for context-free languages. Consider the string $w = a^N b^N c^N$. Suppose $w = uvxyz$, where $|vxy| \leq N$ and $|vy| \geq 1$. If vy contains only a's or vy contains only c's, then pump up: the string $uv^2xy^2z \notin L$. Suppose vy contains only b's. Then we can pump either way to get a string not in L. Suppose v contains two different letters or y contains two different letters. Then uv^2xy^2z is not even of the form $a^*b^*c^*$, so certainly it is not in L. Finally, suppose ($v \in a^+$ and) $y \in b^+$, or $v \in b^+$ (and $y \in c^+$). Then we can pump down and there will be too few b's. By $|vwy| \leq N$, these are all the possible cases. So in all cases there is some i for which $uv^ixy^iz \notin L$, a contradiction.

Problem 5. *Prove that* $L = \{b_i \# b_{i+1} : b_i$ *is* i *in binary,* $i \geq 1\}$ *is not context free.*

Suppose for contradiction that L were context free. Let N be the "N" of the pumping lemma for context free languages. Consider the string $w = 1^N 0^N \# 1^N 0^{N-1} 1 \in L$. Suppose $w = uvxyz$, where $|vxy| \leq N$ and $|vy| \geq 1$. If vxy does not contain a #, then pumping either way will cause a contradiction (increasing or decreasing one of the numbers without touching the other). If the # is contained in v or y, then pumping either way leads to a string not even in $(a \cup b)^* \# (a \cup b)^*$, i.e., a string definitely outside of L. Because of the $|vxy| \leq N$ condition, the only remaining possibility is for $v = 0^i$ and $y = 1^j$, $i, j \geq 1$, to fall on opposite sides of the "#." But pumping up in this case means multiplying the left hand number by some power of two, while it never means multiply the right hand number by some power of two. Thus the pumped-up strings will not remain with the right number one more than the left.