ECS 120: Theory of Computation

Homework 1 Solution

Due: 4/5/06

[Problem 1.] Linz, Page 13, Exercise 1.

Prove by induction on $|S|$.

Basis: When $|S| = 0$, then $S = \emptyset$ and $2^S = \{\emptyset\}$. $|2^S| = 1 = 2^0 = 2^{|S|}$.

Induction Hypothesis: Assume the statement holds for $|S| = k$, such that $|2^S| = 2^k = 2^{|S|}$.

Inductive Step: When $|S| = k + 1$, suppose $S = S_k \cup \{x\}$, where $|S_k| = k$. Then 2^S contains S_k plus the following subsets:

- the sets $\{x\}$ and S
- for each subset of size m in 2^{S_k}, where $1 < m < k$, include x. Thus, the subsets of size m in 2^S now contains every subset of size m in 2^{S_k} and every subset of size $m-1$ in 2^{S_k} to include x (there are C^k_{m-1} extra).

In addition to the set 2^k, we need to add $1 + C^k_1 + C^k_2 + ... + C^k_{k-1} + C^k_k = 2^k$ more subsets (based on binomial series). Thus, $|2^{k+1}| = 2^k + 2^k = 2^{k+1} = 2^{|S|}$.

[Problem 2.] Linz, Page 14, Exercise 7.

Prove both sides of the equation.
If $x \in S_1 \cup S_2$, then $x \notin \overline{S_1 \cup S_2}$. By DeMorgan's Law: $\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}$. Thus, $x \notin \overline{S_1} \cap \overline{S_2} \rightarrow x \in \overline{\overline{S_1} \cap \overline{S_2}}$.

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[Problem 3.] Linz, Page 14, Exercise 10.

Use Boolean Laws for logic.

$$\begin{aligned}
x \in S_1 \cap (S_2 \cup S_3) &\equiv (x \in S_1) \wedge (x \in S_2 \cup S_3) \\
&\equiv (x \in S_1) \wedge ((x \in S_2) \vee (x \in S_3)) \\
&\equiv ((x \in S_1) \wedge (x \in S_2)) \vee ((x \in S_1) \wedge (x \in S_3)) \\
&\equiv (x \in S_1 \cap S_2) \wedge (x \in S_1 \cap S_3) \\
&\equiv x \in (S_1 \cap S_2) \cup (S_1 \cap S_3)
\end{aligned}$$

[Problem 4.] Linz, Page 14, Exercise 14.

Partition the set into equivalence classes based on the relation:

$$x \equiv y \leftrightarrow x \ mod \ 3 = y \ mod \ 3$$

We get $\{6, 9, 24\}$, $\{4, 25, 31, 37\}$, $\{2, 5, 23\}$.

[Problem 5.] Linz, Page 15, Exercise 18.

Given $f(n) = O(g(n)) \Rightarrow f(n) \leq c \cdot g(n)$ and $g(n) = O(h(n)) \Rightarrow g(n) \leq d \cdot h(n)$, where c and d are positive integers.
Then $f(n) \leq cd \cdot h(n)$, thus $f(n) = O(h(n))$.

[Problem 6.] Linz, Page 15, Exercise 25.

Proof by induction on n.

Base case, where $n = 1$

$$\sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{n(n+1)(2n+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1$$

Assume that the statement holds for $n = k$,

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Show that the statement holds for $n = k + 1$.

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Therefore the statement is proven by induction on any $n \geq 1$

[Problem 7.] Linz, Page 15, Exercise 28.

[a.] $f(n) = O(2^n)$

Use induction to show that $f(n) \leq 2^n$.

Basis: when $n = 1, 2$, $f(1) = 1 < 2 = 2^1$ and $f(2) = 1 < 4 = 2^2$.

Inductive hypothesis: suppose $f(k) \leq 2^k$ for $k \geq 3$.

Induction step: show that $f(k+1) \leq 2^{k+1}$.

$$\begin{aligned} f(k+1) &= f(k-1) + f(k) \\ &\leq 2^{k-1} + 2^k = 3 \cdot 2^{k-1} \\ &< 4 \cdot 2^{k-1} = 2^{k+1} \end{aligned}$$

[b.] $f(n) = \Omega(1.5^n)$

Use induction to show that for very large n, $f(n) \geq \frac{1}{3}1.5^n$.

Basis: when $n = 1, 2$, $f(1) = 1 > 0.5 = \frac{1}{3} \cdot 1.5^1$ and $f(2) = 1 > 0.75 = \frac{1}{3} \cdot 1.5^2$.

Inductive hypothesis: suppose $f(k) \geq \frac{1}{3} \cdot 1.5^k$ for $k \geq 3$.

Induction step: show that $f(k+1) \geq \frac{1}{3} \cdot 1.5^{k+1}$.

$$\begin{aligned} f(k+1) &= f(k-1) + f(k) \\ &\geq \frac{1}{3} \cdot (1.5^{k-1} + 1.5^k) = \frac{1}{3} \cdot 2.5 \cdot 1.5^{k-1} \\ &> \frac{1}{3} \cdot 2.25 \cdot 1.5^{k-1} = \frac{1}{3} \cdot 1.5^2 \cdot 1.5^{k-1} = \frac{1}{3} \cdot 1.5^{k+1} \end{aligned}$$

[Problem 8.] Linz, Page 15, Exercise 29.

Proof by contradiction.
Suppose $\sqrt{8}$ is rational, then $\sqrt{8} = 2\sqrt{2} = \frac{a}{b}$ where a and b are integers and have no common factor. This equation gives us $\sqrt{2} = \frac{a}{2b}$. However $\sqrt{2}$ is not rational (proved in Example 1.7), contradiction. Thus $\sqrt{8}$ is not rational.

[Problem 9.] Linz, Page 16, Exercise 32.

[a.] True.

Prove by contradiction, suppose $\frac{a}{b} + irrational = \frac{c}{d}$.
Where $a, b, c, d \in N$
Then $irrational = \frac{bc-ad}{bd}$, contradiction.

[b.] False.

For example, $2 + \sqrt{2} + 2 - \sqrt{2} = 4$.

[c.] True.

Prove by contradiction, suppose $\frac{a}{b} \times irrational = \frac{c}{d}$.
Where a, b, c, d are non-zero integers
Then $irrational = \frac{bc}{ad}$, contradiction.
Thus, the statement is true for non-zero rationals.

[Problem 10.] Linz, Page 16, Exercise 35.

Given any triplet $< n, n+2, n+4 >$ (where $n \geq 2$), the numbers in the triplet must be either all odds or all evens. Here we will focus on only the odd triplets, since we're looking for prime numbers in a triplet. However in every odd triplet, there exists exactly one number that is divisible by 3. By observing the odd number series, $1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, ...$, we can see that between two immediate multiple of 3's, there contains two odd numbers. This means we can pick any consecutive triplet which will contain a mutiple of 3. Therefore the only triplet that contains all primes is $< 3, 5, 7 >$.