Comput. & Elect. Engng, Vol. 3, pp. 409–419. Pergamon Press, 1976. Printed in Great Britain

ON SOLVING FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND USING THE INTERVAL PROGRAMMING ALGORITHM OF ROBERS

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(*Received* 28 *June* 1976)

Abstract—A discretized version of Fredholm integral equation of the first kind is solved using an interval programming algorithm and the results are compared with an initial value method.

1. INTRODUCTION

The Fredholm integral equation of the first kind

$$\int_a^b K(x, y)f(x)\,dx = g(y); \qquad c \le y \le d \tag{1}$$

where f, g and K are real valued functions defined, respectively, on $[a, b] \subseteq R^1$, $[c, d] \subseteq R^1$ and $[c, d] \times [a, b] \subseteq R^2$ occurs in a wide variety of situations involving measurements made with imperfect instruments[1–11]. Typically, the function $g(y)$ would represent the quantity being measured, K is a property of the measuring device and f is the unknown function to be recovered. In an ideal measuring situation, the kernel $K(x, y) = \delta(x - y)$, the delta function; then $f(y) = g(y)$. In practical situations, a broadened function, such as a Gaussian distribution function, takes the place of the delta function. This results in a smearing of f which in turn means a loss of information. The problem of recovering the value of f from observed values of g is therefore not trivial. This nontriviality, which stems from an ill-posedness of the physical problem, translates into nontrivial computational difficulties. For instance, a discretized version of eqn (1) typically tends to be ill-conditioned or underdetermined. Thus, the method described here can also be viewed as one for solving underdetermined systems of linear equations and this has a wide applicability in problems such as the reconstruction of objects from images.

A number of investigators[12–23] proposed a variety of methods to solve eqn (1) and each achieved some measure of success (see Section 2). The purpose of this paper is to demonstrate the feasibility of applying an interval programming (IP) technique to solve eqn (1).

2. PREVIOUS METHODS

Eddington[12] suggested a Taylor series solution which is applicable for the integration range $-\infty$ to ∞ and when $K(x, y)$ is of the form $K(x - y)$. His solution was semi-convergent if the values of the kernel $K(x - y)$ are sharply peaked, and if $f(x)$ and $g(y)$ are relatively smoother than $K(x - y)$. Lane, Moorehouse and Phillips[3] extended this method to the case of a finite integration interval using Fourier series. Tichanov's[13, 14] regularization technique assumes a unique solution $f(x)$ corresponding to an errorless measurement $g(y)$. Assuming that both $f(x)$ and $g(y)$ to be continuous and piecewise smooth, Tichanov determined the function $f(x)$ that minimizes a functional defined as

$$M[f(x), g(y)] = \alpha \int_a^b [q(x)[f'(x)]^2 + p(x)[f(x)]^2]\,dx + \int_a^b \epsilon^2\,dt$$

where $q(x)$ and $p(x)$ are fixed positive functions and α is a parameter. The ϵ in the second term

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on the right is just the residual, i.e. the difference between the left and right sides of eqn (1). The first term on the right is referred to as the regularization functional.

Bellman *et al.*[15] treated eqn (1) using dynamic programming in conjunction with a successive approximation technique. While agreeing that no one technique can satisfactorily resolve the fundamental problem of obtaining sensible solutions from ill-conditioned systems, the authors strived to point out that a reasonable compromise is to combine various techniques in optimal manner. In a discussion of the difficulties involved in solving eqn (1), Phillips[16] demonstrated that the existence of errors converts the problem from one with a unique solution to one with infinitely many solutions. To pick one, out of many solutions, it is necessary to impose constraints. Phillips picks the solution that minimizes $\int_a^b [f''(x)]^2\,dx$ over the family of solutions. This type of smoothing can also be achieved using other types of criteria. Following Phillips' lead, Towmey[17] formulated the problem as a constrained optimization problem using Lagrange multipliers. One common feature of the preceeding methods is that they require a matrix to be inverted which is, in general, a time consuming operation. To overcome the storage and time restrictions, Hunt used fast Fourier transforms. This method lacks the generality of Phillips–Towmey method. Recently, the author[18, 19] suggested an initial value formulation which converts the original problem into another requiring solution of a system of ordinary differential equations with prescribed initial conditions. This method was implemented on both analog and digital computers with promising results.

3. DISCRETIZATION

A standard method of solving eqn (1) is to replace the continuous variable y with a finite set of mesh points $y_1, y_2, \ldots, y_n$ such that $a \le y, < y_2 < y_3 \ldots < y_n < b$ and to write

$$\begin{aligned} g(y_1) &= \int_a^b K(x, y_1) f(x)\,dx \\ &\vdots \qquad\qquad \vdots \\ g(y_m) &= \int_a^b K(x, y_m) f(x)\,dx. \end{aligned}$$

In experimental situations $g(y_i) \triangleq g_i$, $i = 1, 2, \ldots, m$, generally correspond to m experimentally observed data points.

The next step of the discretization process is to replace the continuous variable x by a finite set of mesh points $x_1, x_2, \ldots, x_n$ such that $a \le x_1 < x_2 < \ldots < x_n \le b$.

To carry out this step, each of the integrals in eqn (2) is replaced by a suitably chosen numerical quadrature formula to yield

$$\begin{aligned} g(y_1) &\approx \sum_{j=1}^{n} w_j K(x_j, y_1) f(x_j) \\ &\vdots \qquad\qquad \vdots \\ g(y_m) &\approx \sum_{j=1}^{n} w_j K(x_j, y_m) f(x_j) \end{aligned}$$

where $w_1, w_2, \ldots, w_n$ are the weights associated with the quadrature formula. Equation (3) in vector-matrix notation is

$$g \approx Bf$$

where $g = [g(y_1), g(y_2), \ldots, g(y_m)]^T$, $f = [f(x_1), f(x_2), \ldots, f(x_n)]^T$ in which the superscript T denotes the transpose and B is an $m \times n$ matrix given by

$$B = \begin{bmatrix} w_1 K(x_1, y_1) & w_2 K(x_2, y_1) & \ldots & w_n K(x_n, y_1) \\ w_1 K(x_1, y_2) & w_2 K(x_2, y_2) & \ldots & w_n K(x_n, y_2) \\ \vdots & & & \\ w_1 K(x_1, y_m) & w_2 K(x_2, y_m) & \ldots & w_n K(x_n, y_m) \end{bmatrix}. \tag{5}$$

The solution of the integral eqn (1) is thus equivalent to the solution of the matrix eqn (4). There are many possible sources for error in experimental and computational processes: inaccuracies in the original recorded data, truncation errors resulting from quadrature approximations, rounding errors caused due to the finite word length of digital computers. To accommodate at least some of these errors, eqn (4) is rewritten as

$$g = Bf + \epsilon \tag{6}$$

where $\epsilon \in R^m$ is a vector of errors in the measured values of g.

Solution of eqn (6) is beset with difficulties primarily because of the poor conditioning of matrix B. While eqn (1) is ill-posed, the matrix B in eqn (6) is ill-conditioned. Equation (1) is sensitive to measurement errors and sampling errors. If one attempts to choose m and n larger and larger in an effort to avoid discretization or sampling errors, the condition of B will correspondingly deteriorate. It can be shown that as $m \to \infty$ and $n \to \infty$, then B indeed approaches the Hilbert matrix, which is often cited as a classical example of an ill-conditioned matrix. That is to say that large n tends to make B more and more ill-conditioned. Making n too small is not desirable either as it tends to make the approximation very crude.

The basic difficulty with ill-posed problems is the lack of sufficient information from measurements to infer the correct solution. This is reflected mathematically in the fact that the system determinant B tends to be underdetermined, or rank deficient, even if it is formally overdetermined (i.e. more rows than columns). An obvious approach, then, is to augment the data with any additional knowledge of the nature of the quantity being measured in order to obtain approximate solutions which are physically meaningful. One method of accomplishing this objective is to translate some of the physical insight into additional mathematical constraints. For instance, the error vector ϵ is usually composed of both measurement errors, and discretization and rounding errors associated with the computation. In well designed experiments that utilize well calibrated instruments, the errors vector can be assumed to have zero mean. This can be translated into the constraint $-\epsilon_b \le \epsilon \le +\epsilon_b$ where ϵ_b is a measure of the spread of error distribution. Therefore, the task of solving eqn (1), or eqn (5), can be posed as an L_1 approximation problem:

$$\begin{aligned} &\text{Minimize} \quad \sum_{i=1}^{n} |\epsilon_i| \\ &\text{Subject to} \quad Bx + \epsilon = g, \quad -\epsilon_b \le \epsilon \le \epsilon_b. \end{aligned} \tag{7}$$

If matrix B has more rows than columns (i.e. $m > n$), then it is advantageous to work with the dual problem

$$\begin{aligned} &\text{Maximize} \quad t^T y \\ &\text{Subject to} \quad B^T y = 0, \quad -e \le y \le e \end{aligned}$$

where the vector e is defined as $e = (1, 1, \ldots, 1)^T$ for program standardization. This operation may require a scaling of the original equations.

4. THE INTERVAL PROGRAMMING ALGORITHM

Equation (7) can be rewritten in the standard interval programming (IP) format as

$$\text{maximize} \quad t^T y \tag{8a}$$

$$\text{subject to} \quad b^- \le Ay \le b^+ \tag{8b}$$

where

$$A = \begin{pmatrix} B^T \\ I \end{pmatrix}; \quad b^- = \begin{pmatrix} 0 \\ -e \end{pmatrix}; \quad b^+ = \begin{pmatrix} 0 \\ +e \end{pmatrix}.$$

Robers and Ben-Israel [24, 25] suggested an efficient supoptimization algorithm for solving the IP

problem posed in eqn (8). This method, which is finite-iterative in nature, attempts to solve eqn (8) in terms of an auxilary problem. For the νth iteration ($\nu \geq 1$), this auxilary problem takes the form:

$$\text{maximize} \quad t^T y \tag{8a}$$

$$\text{subject to} \quad b^{(\nu)-} \leq A^{(\nu)} y \leq b^{(\nu)+} \tag{8c}$$

$$b^{h(\nu)-} \leq a^{h(\nu)} y \leq b^{h(\nu)+} \tag{8d}$$

where eqn (8c) is a set of n constraints from eqn (8b) chosen such that $A^{(\nu)}$ is nonsingular and eqn (8d) is a single constraint equation arbitrarily picked from the remaining $(m-n)$ equations of eqn (8b). Thus, eqns (8a, c, d) constitute a subproblem of the one defined by eqns (8a, b).

The calculations to be performed at the νth iteration are now described below. Let

$$y^{(\nu-1)} = \text{optimal solution of eqns (8a, c)}$$

$$y^{(\nu)} = \text{optimal solution of eqns (8a, c, d)}.$$

To start the process, let, for $\nu = 1$

$$A^{(1)} = I, \quad b^{(1)-} = -e, \quad b^{(1)+} = e \tag{9}$$

and let eqn (8d) be any constraint from the remaining constraints $0 \leq B^T y \leq 0$ of eqn (8b). Then, the starting solution $y^{(0)} = (y_1^{(0)}, \ldots y_u^{(0)})^T$ where

$$y_i^{(0)} = \begin{cases} 1 & \text{if } t_i > 0 \\ -1 \leq \theta_i \leq +1 & \text{if } t_i = 0 \\ -1 & \text{if } t_i < 0 \end{cases}; \qquad i = 1, 2, \ldots, n$$

is clearly a solution of the subproblem

$$\begin{aligned} &\text{maximize} \quad t^T y \\ &\text{subject to} \quad -e \leq y \leq e. \end{aligned} \tag{11}$$

Thus, the procedure can get started with a feasible solution. For $\nu \geq 1$, the values of $y^{(\nu-1)}$ and $[A^{(\nu)}]^{-1}$ are already known. Now, if $y^{(\nu-1)}$, the optimal solution of eqns (8a, c), satisfies eqn (8d) also, then by the earlier definition

$$y^{(\nu)} = y^{(\nu-1)}. \tag{12}$$

Otherwise, $y^{(\nu)}$ can be obtained from $y^{(\nu-1)}$ as follows: Let Δ, defined below, be the amount by which eqn (8d) is violated by $y^{(\nu-1)}$

$$\Delta = \begin{cases} a^{h(\nu)} y^{(\nu-1)} - b^{h(\nu)+} & \text{if positive} \\ a^{h(\nu)} y^{(\nu-1)} - b^{h(\nu)-} & \text{if negative.} \end{cases} \tag{13}$$

Now let

$$Q = \{i : 1 \leq i \leq n, (a^{h(\nu)}(A^{(\nu)})^{-1})_i \neq 0, \gamma_i \geq 0\}, \tag{14}$$

where

$$\gamma = \frac{(t^T (A^{(\nu)})^{-1})_i}{(a^{h(\nu)}(A^{(\nu)})^{-1})_i} (\text{sign } \Delta), \tag{15}$$

be an index set of the components of $(a^{(\nu)} y^{(\nu-1)})$. Notice that Q is the set of components of $(A^{(\nu)} y^{(\nu-1)})$ which can be changed in order to move eqn (8d) to feasibility while maintaining feasibility in eqn (8c) and γ_i can be interpreted as the marginal cost of such a change.

Now, the indices in Q are ordered such that

$$Q = (k_1, k_2 \ldots k_q) \tag{16}$$

where

$$\gamma_{k_1} \leq \gamma_{k_2} \leq \gamma_{k_3} \leq \ldots \leq \gamma_{k_q}. \tag{17}$$

If $\gamma_{k_i} = \gamma_{k_{i+1}}$ then as an arbitrary tie-breaking rule, it is proposed to set $k_i < k_{i+1}$.

Now a modified delta is defined as

$$\delta_k = \begin{cases} (b^{(\nu)-} - A^{(\nu)} y^{(\nu-1)})_k & \text{if sign } \Delta = \text{sign } (a^{h(\nu)}(A^{(\nu)})^{-1})_k \\ (b^{(\nu)+} - A^{(\nu)} y^{(\nu-1)})_k & \text{if sign } \Delta = -\text{sign } (a^{h(\nu)}(A^{(\nu)})^{-1})_k \end{cases} \tag{18}$$

for $k \in Q$. Introducing

$$p = \min \left[i : 1 \leq i \leq q, \left| \sum_{j=1}^{i} \delta_{k_j} (a^{h(\nu)}(A^{(\nu)})^{-1})_{k_j} \right| \geq |\Delta| \right] \tag{19}$$

and

$$\theta = \frac{-\Delta - \sum_{j=1}^{p-1} \delta_{k_j} (a^{h(\nu)}(A^{(\nu)})^{-1}) k_j}{(a^{h(\nu)}(A^{(\nu)})^{-1})_{k_p}}. \tag{20}$$

Now $y^{(\nu)}$ can be expressed in terms of $y^{(\nu-1)}$

$$y^{(\nu)} = y^{(\nu-1)} + (A^{(\nu)})^{-1} \left[\sum_{i=1}^{p-1} \delta_{k_i} e_{k_i} + \theta e_{k_p} \right] \tag{21}$$

where e_k is the kth unit vector.

Here δ_k is the maximal change of $(A^{(\nu)} y^{(\nu-1)})_k$ in moving eqn (8d) towards feasibility without violating eqn (8c). In effect, $y^{(\nu)}$ is obtained from $y^{(\nu-1)}$ by making p such changes. If the $y^{(\nu)}$ thus obtained satisfies all the constraints of eqn (8b), then the problem is solved. In going from νth to $(\nu+1)$th iteration, the k_pth constraint of eqn (8c) is replaced by eqn (8d). In place of eqn (8d), any constraint from eqn (8b) that is violated by $y^{(\nu)}$ is used. The algorithm terminates after a finite number, say f, of iterations and $y^{(f)}$ is the optimal solution of the auxilary problem defined by eqns (8a, c and d). This optimal solution $y^{(f)}$ can now be used to obtain the optimal solution ϵ^* and x^* of eqn (6) as follows.

At the end of the final iteration, $A^{(f+1)}$ is obtained from $A^{(f)}$ as a result of replacing the latter's k_pth row with $a^{h(f)}$. Then

$$x_j^* = \begin{cases} 0 \text{ if row } j \text{ of } B^T \text{ is not in } A^{(f+1)} \\ t^T (A^{(f+1)})^{-1})_i \text{ if row } j \text{ of } B^T \text{ is the } i\text{th row of } A^{(f+1)} \end{cases} \tag{22}$$

and

$$e^* = t - Bx^*. \tag{23}$$

5. ILLUSTRATIVE EXAMPLES

The interval programming algorithm described in the preceeding section was applied to solve several problems [26] and the results were compared with those obtained by other methods. Some of the more interesting results are presented in this section. In all cases, the starting point was an equation such as eqn (1). The matrix B in eqn (5) was obtained by using Simpson's quadrature at eqn (3).

In all cases, the right side of eqn (1), namely $g(y)$ is assumed to be known and one is required

to find $f(x)$. Therefore, the values of $g(y_i)$, $i = 1, \ldots, m$, are first evaluated and their information is corrupted with a random noise ϵ whose mean value can be changed as a parameter. To facilitate comparison, only simple problems for which $f(x)$ can be exactly evaluated are considered. Thus, the function to be recovered from observational data is indeed known *a priori.* The computational algorithm produces a set of points $f(x_i)$. Then an interpolating polynomial $\phi(x)$ is fitted to the above set of points.

Example 1

Consider

$$\int_{0.5}^{5.5} K(x, y) f(x)\, dx = g(y); \qquad 3 \leq y \leq 7 \tag{24}$$

with

$$k(x, y) = e^{-(x-y)^2} \tag{25}$$

$$f(x) = 1/3\, x^3 - 3x^2 + 8x + 1. \tag{26}$$

This problem was discussed earlier by Vemuri[19]. If one performs the integration indicated in eqn (1), one readily gets an expression for $g(y)$. This was evaluated for various values of y_i and was used as data to recover $f(x_i)$ for all i. To facilitate comparison, an interpolating polynomial $\phi(x)$, defined by,

$$\phi(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \tag{27}$$

was fitted to the set of points $f(x_i)$ and the c_j are compared with the coefficients in eqn (27).

Figure 1 shows a faithfully recovered solution $f(x_i)$ when no errors were introduced. Although, Fig. 1 shows an apparently perfect fit, the polynomial coefficients c_j are not the same. Table 1 shows this comparison.

Figure 2 shows the behavior of the solution $f(x)$ as the errors ϵ are introduced and increased and Fig. 3 shows the behavior of the solution as the number m of constraints are varied. Figure 4

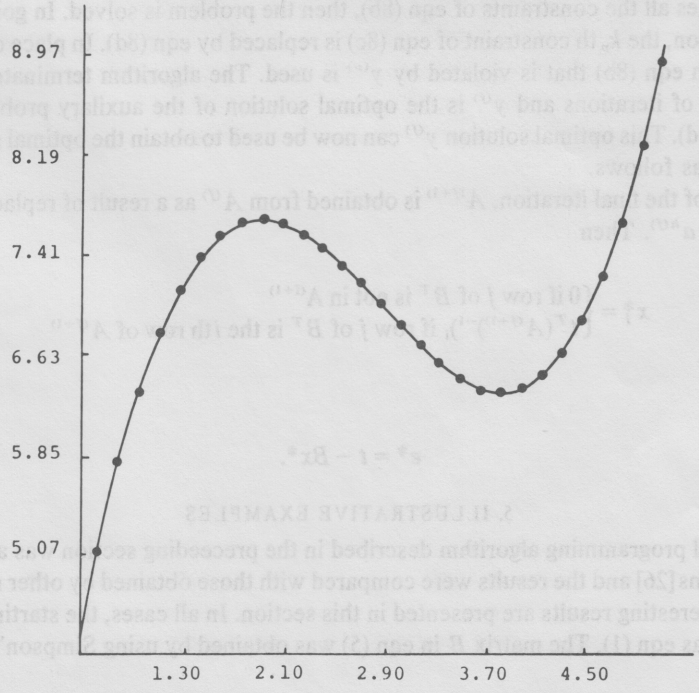

Fig. 1. Solution of eqn (24) using the interval programming algorithm.

Table 1.

Coefficient	Original value	IP solution
c_1	1.00 E + 00	7.00 E + 00
c_2	8.00 E + 00	− 2.5 E + 00
c_3	− 3.00 E + 00	− 1.56 E − 12
c_4	3.33 E − 01	5.21 E + 00

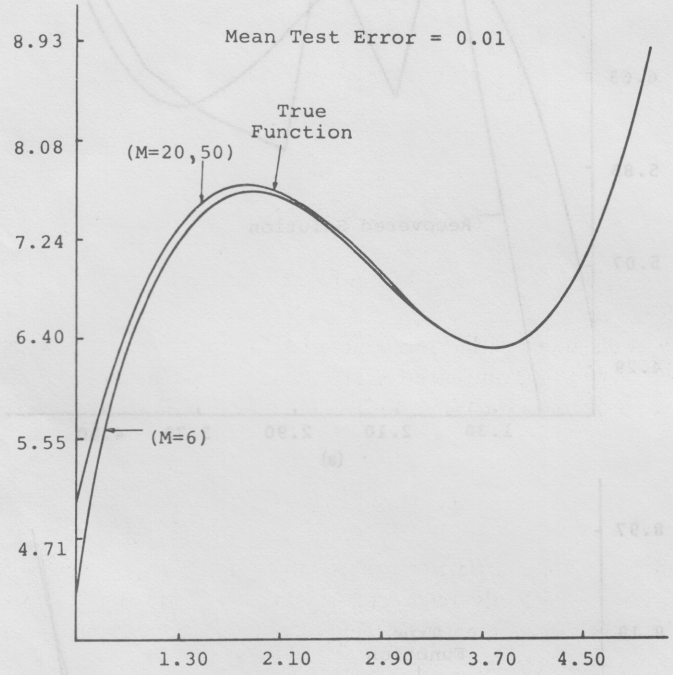

Fig. 2. Solution of eqn (24) with varying number of constraints and with a mean observational error = 0.01.

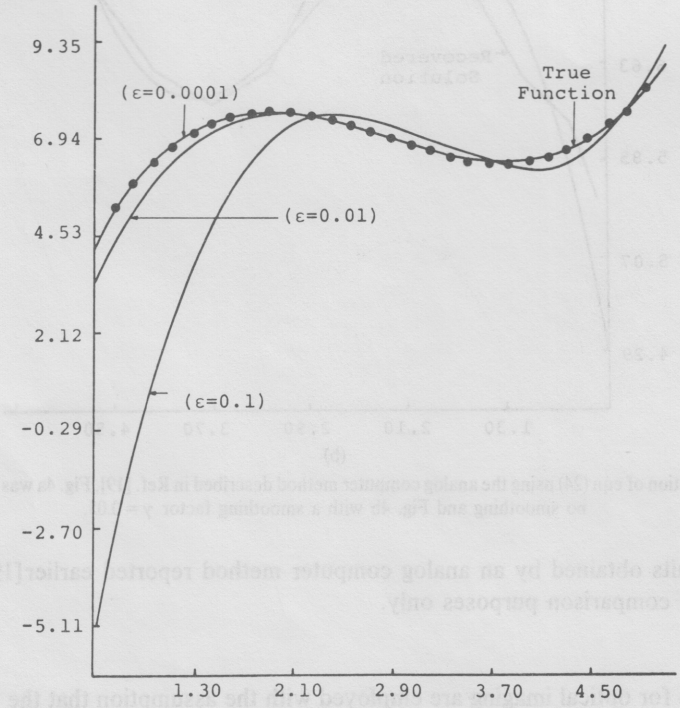

Fig. 3. Solution of eqn (24) with varying amount of mean observational error.

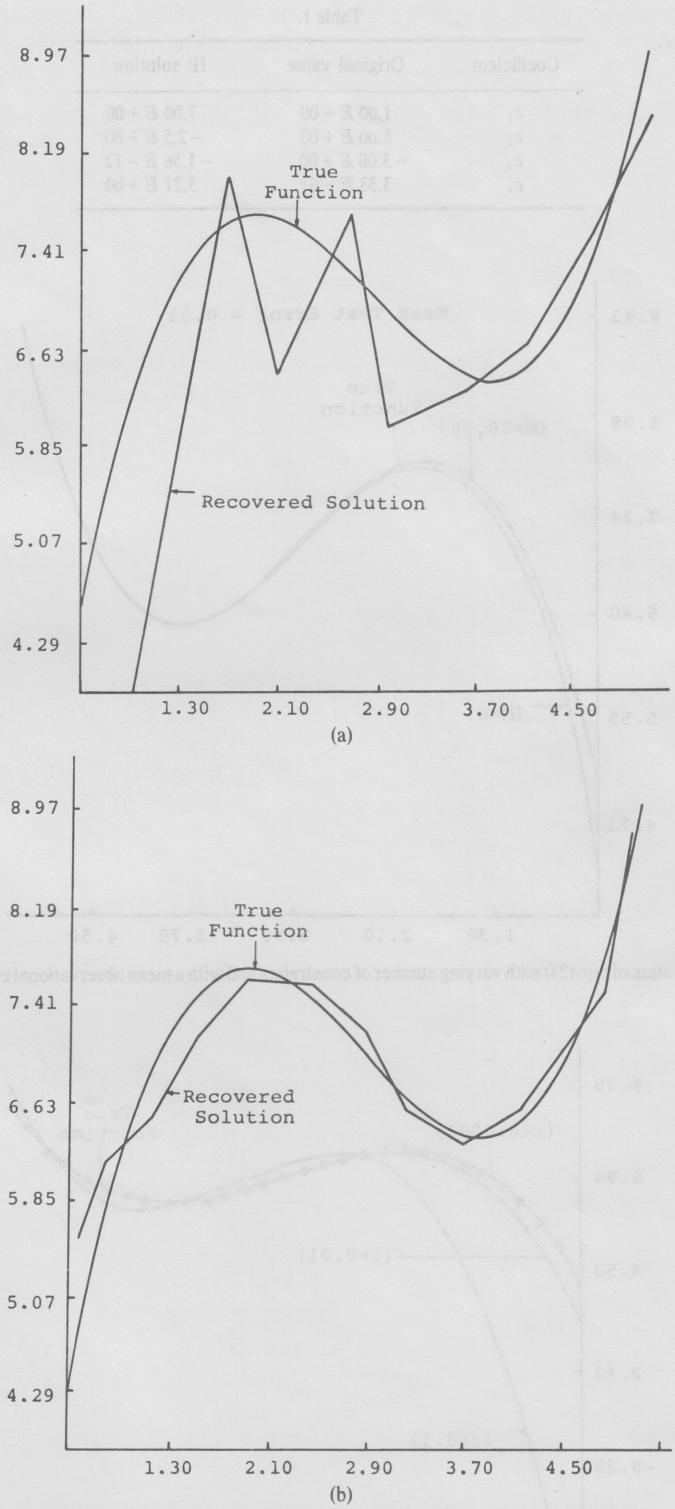

Fig. 4. Solution of eqn (24) using the analog computer method described in Ref. [19], Fig. 4a was obtained with no smoothing and Fig. 4b with a smoothing factor $\gamma = 0.05$.

shows the results obtained by an analog computer method reported earlier [19]. This figure is reproduced for comparison purposes only.

Example 2

Instruments for optical imaging are employed with the assumption that the intensity at each point y of the image depends solely on the intensity of a single point x of the source. After

observing the intensity $g(y)$ throughout the image, one can invert eqn (1), once again, to get the source function $f(x)$. Calibrating the instrument corresponds to a mathematical formulation of the kernel $K(x, y)$.

Consider an optical device with a kernel defined by

$$K(x, y) = K(\phi, \theta) = [\cos\phi + \cos\theta)((\sin\alpha))\alpha)]^2$$

with

$$\alpha = (\pi a/\lambda)(\sin\phi + \sin\theta).$$

The angle of incidence ϕ defines the location of the source and the angle of emergence defines the location of the image points. Traditionally, an undeviated ray is assumed to be described by $\theta = -\phi$ and the slit width consists of only one wavelength, i.e. $a = \lambda$. In practical applications, the objects mostly examined had the form of superimposed Gaussians

$$f(x) = x(\phi) = \exp(-c_1(\phi - d_1)^2) + k\exp(-c_2(\phi - d_2)^2)$$

with $k = 1$, c_1 and c_2 as high as 16, and d_1 and d_2 as small as $\pm 1/4$. Shaw [27] solved this problem using the set of parameters

$$\begin{array}{lll} c_1 = c_2 = 4 & -1.5 \le \phi \le 1.5 & k = 1 \\ d_1 = d_2 = 0.5 & -\pi/2 \le \theta \le \pi/2 & \end{array} \quad . \tag{28}$$

The above problem was solved once again using the IP algorithm. Shaw's original result is shown in Fig. 5 and the IP solution in Fig. 6.

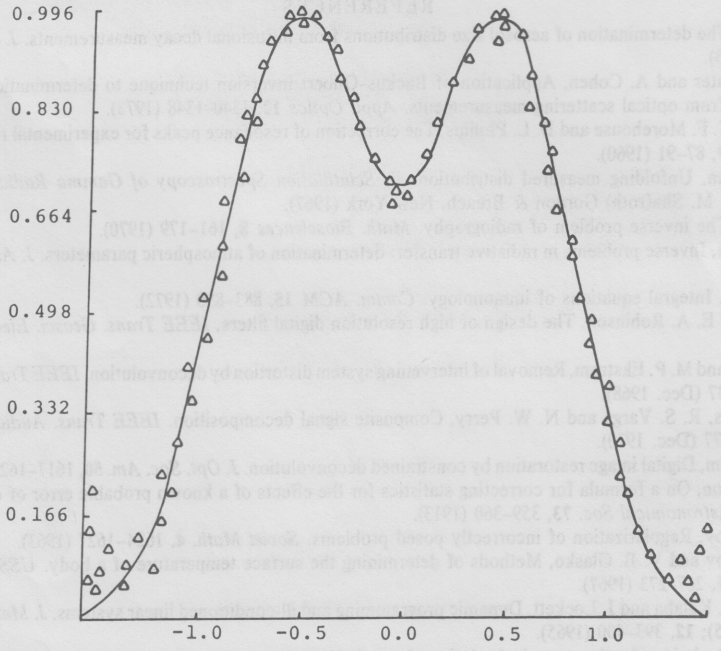

Fig. 5. Solution of the image restoration problem using Shaw's method.

DISCUSSION

This paper essentially demonstrated the feasibility of solving some types of Fredholm integral equation of the first kind using interval programming. The accuracy of the method is not readily apparent from the graphs but the computer output revealed that the method is at least as good as most of the available methods cited in the earlier part of the paper. Further comparitive studies would be useful in establishing the relative merits of various methods in different circumstances.

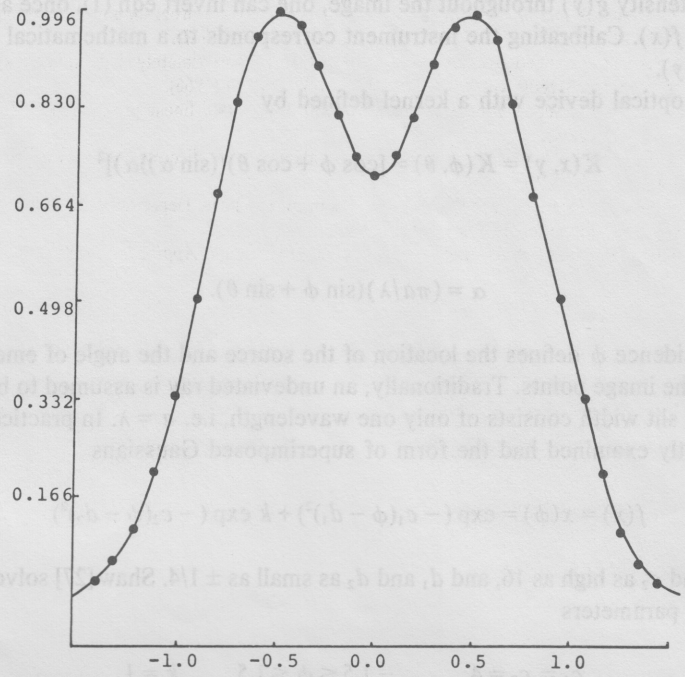

Fig. 6. Solution of the image restoration problem using the interval programming method.

Acknowledgement—Problems presented in this paper were programmed and solved by the second author as a part of his M.S. thesis done under the supervision of the first author. The authors appreciate the help of Dr. Philip Robers of the Research Analysis Corporation for his assistance.

REFERENCES

1. S. Towmey, The determination of aerosol size distributions from diffusional decay measurements. *J. Franklin Inst.* **275**, 121–138 (1963).
2. E. R. Westwater and A. Cohen, Application of Backus–Gilbert inversion technique to determination of aerosol size distributions from optical scattering measurements. *Appl. Optics* **12**, 1340–1348 (1973).
3. R. O. Lane, N. F. Morehouse and D. L. Phillips, The correction of resonance peaks for experimental resolution. *Nuclear Instr. Meth.* **9**, 87–91 (1960).
4. J. E. Monahan, Unfolding measured distributions in *Scintillation Spectroscopy of Gamma Radiation* pp. 371–428. (Edited by S. M. Shafroth) Gordon & Breach, New York (1967).
5. B. R. Hunt, The inverse problem of radiography. *Math. Biosciences* **8**, 161–179 (1970).
6. M. T. Chahine, Inverse problems in radiative transfer: determination of atmospheric parameters. *J. Atmospheric Sci.* **27** (1970).
7. R. J. Hanson, Integral equations of immunology. *Comm. ACM* **15**, 883–890 (1972).
8. S. Treitel and E. A. Robinson, The design of high resolution digital filters, *IEEE Trans. Geosci. Electron* **GE-4**, 25–38 (June 1966).
9. R. L. Rhoads and M. P. Ekstrom, Removal of intervening system distortion by deconvolution. *IEEE Trans. Instrum. Meas.* **IM-17**, 333–337 (Dec. 1968).
10. D. G. Childers, R. S. Varga and N. W. Perry, Composite signal decomposition. *IEEE Trans. Audio Electro-accoust.* **AU-18**, 471–477 (Dec. 1970).
11. D. P. MacAdam, Digital image restoration by constrained deconvolution. *J. Opt. Soc. Am.* **50**, 1617–1627 (Dec. 1970).
12. A. S. Eddington, On a formula for correcting statistics for the effects of a known probable error of observation. *Mhly Notices, R. Astronomical Soc.* **73**, 359–360 (1913).
13. A. N. Tichanov, Regularization of incorrectly posed problems. *Soviet Math.* **4**, 1624–1627 (1963).
14. A. N. Tichanov and V. B. Glasko, Methods of determining the surface temperature of a body. *USSR Comput, Math. Math. Phys.* **7**, 267–273 (1967).
15. R. Bellman, R. Kalaba and J. Lockett, Dynamic programming and ill-conditioned linear systems. *J. Math. Anal. Appl.* **10**, 206–215, (1965); **12**, 393–400 (1965).
16. D. Phillips, A technique for the numerical solution of certain integral equations of the first kind. *J. ACM* **9**, 84–97 (1962).
17. S. Towmey, On the numerical solution of Fredholm integral equations of the first kind by the inversion of the linear system produced by quadrature. *J. ACM* **10**, 97–101 (1963).
18. V. Vemuri and F. P. Chen, An initial value method for solving Fredholm integral equations of the first kind. *J. Franklin Inst.* **297**(3), 187–200 (Mar. 1974).
19. V. Vemuri, An analog computer method to solve Fredholm integral equations of the first kind. *Comput. Elect. Engng.* **2**, 95–103 (1975).
20. J. Douglas, Mathematical programming and integral equations: *Symposium on the Numerical Treatment of Ordinary Differential Equations, Integral and Integro-Differential Equations*, PICC Rome, Birkhauser-Berlag, pp. 269–274 (1960).
21. I. Barrodale and A. Young, Algorithms for best L_1 and L_∞ linear approximations on a discrete set. *Numer. Math.* **8**, 295–306 (1966).

22. J. Replogle, B. D. Holcomb and W. R. Burrus, The use of mathematical programming for solving singular and poorly conditioned systems of equations. *J. Math. Anal. Appl.* **20**, 310–324 (1967).
23. J. Rosen *et al.*, Talks on generalized inverses and solutions of large, approximately singular linear systems. Mathematical Research Center Technical Report No. 644, University of Wisconsin (1966).
24. P. D. Robers and A. Ben-Israel, A suboptimization method for interval linear programming. *Linear Algebra and Its Applications* **3**, 383–405 (1970).
25. P. D. Robers and A. Ben-Israel, An interval programming algorithm for discrete linear L_1 approximation problems, *J. Approx. Theory* **4**, 323–336 (1969).
26. P. C. Tapia, The ill-posed problem of system identification. M.S. Thesis, Department of Aeronautics & Astronautics, Purdue University, W. Lafayette, IN 47907 (May 1974).
27. C. B. Shaw, Numerical solution of an integral equation. *J. Math. Anal. Appl.* **37**, 83–112 (1972).