Identification of Nonlinear Parameters of Ground Water Basins by Hybrid Computation

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Abstract. Identification of parameters of an unconfined aquifer in which the dynamics of the water table are describable by a partial differential equation can be looked upon as a control system problem in distributed parameter systems. Using a maximum principle in conjunction with a steepest descent algorithm, the transmissibility of an aquifer is identified, starting from observed values of input-output as data. This algorithmic procedure is blended with a heuristic method to identify the storage coefficient and the boundary of an aquifer. Results of computations carried out on a hybrid computer are presented.

INTRODUCTION

Problems encountered in hydrologic systems analysis can be broadly classified as direct and inverse problems. If a system is visualized as a black box with a set of input-output terminals, then direct problems are characterized by a complete specification of the contents of the black box, and one determines the response of the system for any specified input. In contrast to this, an inverse problem consists of a specification of a set of input-output records, from which the contents of the black box are required to be determined. Stated in such general terms, there is no unique solution to the inverse problem, and selection of one solution out of a multitude of possible solutions is largely governed by physical considerations and constraints. For instance, in the case of a design problem, constraints such as physical realizability, reliability, and economy determine the basis for selecting one out of several possible alternatives. On the other hand, in the identification problem, where one is interested in the mathematical characterization of an existing physical system, the corresponding constraints really amount to the selection of the mathematical model that has the greatest probability of characterizing the conditions actually existing in the system under study. The identification problem is therefore invariably guided by a profound understanding of the system under study.

The difficulty of studying systems whose dynamics are characterized by partial differential equations (PDEs) in more than one space dimension lies not only in the complexity of the problem but also in the fact that it is virtually impossible to express all the constraints in an algorithmic form. This bottleneck has severely limited the utility of the conventional computer approach to complex inverse problems. The hybrid computer method [*Vemuri and Dracup,* 1967] can make an important contribution by permitting powerful and useful computational algorithms to be programmed on the digital computer, while allowing the specialist to introduce his judgments and insights as they are required in the computational process, without first translating them into an abstract programming language [*Karplus and Vemuri,* 1967].

Several useful inverse problems leading to parameter identification occur in hydrology. In the context of recent campaigns to conserve natural resources, attention is being shifted from the management of one resource or one aspect of one resource to the management of the total environmental system. Efficient operation of various environmental entities conjunctively to yield best results depends heavily upon the interaction of the individual systems operating on

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the same resource of aiming for the same kind of objective. To this end, a quantitative knowledge of the dynamic behavior of the individual entities is imperative. Only such a knowledge will provide the ability to control and predict the interactions among the individual entities of the total system.

In mathematical language, one says that a specification of a system is necessary to predict and control its behavior. In other words, a system—that is, its parameters—should be identified before one can control it. Treatment of an example illustrating the identification of parameters of a system whose observed input-output records are available would be an effective vehicle to develop the hybrid computer method of parameter identification.

Ground-water Management

An important engineering problem in ground-water hydrology concerns the determination of aquifer parameters from a limited number of observations. For the purpose of formulating an equation, a ground-water basin may be regarded as a three-dimensional field with a free boundary (i.e., the water table) at the upper surface. Designating the elevation of water table above a reference level as $h = h(x, y, t)$, its dynamic state under certain assumptions can be described by a quasilinear parabolic PDE [*Vemuri and Dracup*, 1967]

$$\frac{\partial}{\partial x}\left[T(x, y, h)\frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial y}\left[T(x, y, h)\frac{\partial h}{\partial y}\right] = S(x, y, h)\frac{\partial h}{\partial t} - Q(x, y, t) \qquad (1)$$

defined over a connected region R, the areal extent of the aquifer. Whether or not this equation really represents a ground-water flow field depends upon the validity of the assumptions made in deriving it. Assuming that this equation does indeed characterize a ground-water flow field, the problem is to determine $T(x, y, h)$, $S(x, y, h)$ and the boundary ∂R of the region R such that the observed water table elevations $\hat{h}$ and the response h of the computer model are close to each other in some acceptable sense at least at the points of a finite-difference grid. It is important to remember that the goal is *not* to determine T and S as functions of the continuous variables x, y, and h, but only to approximate their values numerically at those points defined by the finite difference grid.

Conventionally, in a wide class of diffusion processes it is customary to amalgamate the parameters T and S into a single parameter, called diffusivity. Determination of this single parameter from one equation and one set of input-output records is relatively straightforward. However, from an engineering viewpoint, it is more useful to determine T and S separately. Venturing to determine all the three unknowns, viz., T, S, and ∂R, from one equation which is only of first order in its parameters, may lead to meaningless results. It is precisely here that a specialist has a chance to infuse his experience heuristically into the precisely formulated description of a physical phenomenon.

This paper demonstrates how the powerful algorithmic techniques [*Bryson and Denham*, 1962; *Rozonoer*, 1960] available to control systems engineers can be successfully blended with a heuristic approach to arrive at the most plausible computer model of a ground-water basin. To this end, a steepest descent algorithm is developed to identify the parameter T; the other two parameters, i.e., S and ∂R, are identified heuristically.

This identification, termed hybrid identification, is achieved by first assuming a nominal shape to ∂R and a nominal set of values to S. For this setup, the quasilinear PDE describing the dynamic state of the system is linearized piecewise in the time domain. In each temporal subinterval, the dynamics of the system are then described by a linear PDE which is of first order in the only unknown parameter, namely T. This single unknown parameter is identified in a conventional way by minimizing an integral square error cost functional over the T-parameter space. Application of the maximum principle [*Rozonoer*, 1960] guarantees that minimization of an integral cost functional is equivalent to extremizing a suitably defined scalar valued Hamiltonian in Hilbert space. A set of necessary conditions in the form of a pair of PDEs to minimize (or maximize) this Hamiltonian is derived, which in turn will determine optimum estimates of transmissibility.

The search for a coarse extremum of the Hamiltonian is performed by a steepest descent (or ascent) method in Hilbert space. The pair of canonical equations, termed the dynamic and adjoint PDEs, are solved using finite-difference methods on a hybrid computer. The fine structure of the system is revealed by iteratively applying the steepest descent algorithm punctuated with a manual adjustment of the parameters S and ∂R. Changes in S are accomplished by changing a suitable number in the computer memory via the console. If a change in the boundary geometry is warranted, it can be accomplished by changing a few wires on the easily accessible analog patch board of the hybrid computer.

THE IDENTIFICATION ALGORITHM

The Maximum Principle

A common problem in control theory is to determine a control law $c(t)$ such that a differential equation

$$dx/dt = f[x(t), c(t)] \tag{2}$$

representing the dynamic state of a system is satisfied, whereas a cost functional

$$J = \int_{t_i}^{t_f} L(x, c)\, dt \tag{3}$$

is minimized over the interval $[t_i, t_f]$.

The maximum principle states that minimization of the integral J over the interval $[t_i, t_f]$ is equivalent to minimization (or maximization) of a scalar-valued Hamiltonian H, where H is defined by

$$H(x, t, c, v) = v(t)\cdot f(x, c) + L(x, c) \tag{4}$$

in which the multiplier $v(t)$ is called the adjoint of the variable $x(t)$.

If $c(t)$ is treated as a parameter, then under certain conditions the above problem can be treated as a parameter identification problem [*Butkowskii and Lerner,* 1960]. In this section a maximum principle for this identification problem is derived by using the principle of optimality and dynamic programming [*Bellman,* 1957; *Wang,* 1964].

Equation 1 is quasilinear, whereas the identification technique developed in the sequel applies only to linear systems. To be able to apply the theory developed in the succeeding paragraphs, equation 1 is linearized piecewise in the time domain. For this purpose, the time interval $[t_i, t_f]$ over which the equation is required to be solved is subdivided into k subintervals, and it is assumed that T and S are only functions of the spatial coordinates and independent of the variable h during each subinterval. Then equation 1 can be rewritten as

$$Lh(x, y, t) = f(x, y, h, Q) \tag{5}$$

for

$t \in [t_j, t_{j+1}]$

$x, y \in R$ and $j = 0, 1, 2, \cdots, (k-1)$

In Equation 5, the operator L is defined as

$$L = \sigma\left\{\frac{\partial}{\partial x}\left[T(x, y)\frac{\partial}{\partial x}\right] + \frac{\partial}{\partial y}\left[T(x, y)\frac{\partial}{\partial y}\right]\right\} \tag{6}$$

and

$$f = (\partial h/\partial t) - \sigma Q(x, y, t) \tag{7}$$

where $\sigma(x, y) = 1/S(x, y)$.

Now the identification problem in its continuous form can be stated as follows: Given the observed values $\hat{h}(x, y, t)$, find T, S, and ∂R such that

$$J = \int_{t_j}^{t_{j+1}} \int_R [h(x, y, t) - \hat{h}(x, y, t)]^T \cdot [h(x, y, t) - \hat{h}(x, y, t)]\, dR\, dt \tag{8}$$

is minimized for each j, and the resulting values of T, S, and ∂R are plausible from physical considerations. The superscript T stands for transpose.

In the present example h is a scalar, and the quantity $[h(x, y, t) - \hat{h}(x, y, t)]^T[h(x, y, t) - h(x, y, t)]$ appearing under the integral sign of equation 8 becomes $(h(x, y, t) - \hat{h}(x, y, t))^2$, but the expression as it appears in equation 8 is written to include the general case where h is an n-dimensional vector valued function. The important thing to remember is that by equation 8 we seek to minimize the square of the distance between two continuous functions, and so the distance is to be measured in function space.

Let the minimum value of J over a subinterval $[t_0, t_1]$ and over all possible values of $T(x,y,t)$ be defined as

$$\Pi[h(t_0, x, y), \tau] \triangleq \min_{\substack{T(t,x,y)\\ t\in[t_0,t_1]}} J \tag{9}$$

$$= \min_{\substack{T(t,x,y)\\ t\epsilon[t_0,t_1]}} \int_{t_0}^{t_1} \int_R (h - h)^T(h - \hat{h})\, dR\, dt \quad (10)$$

Obviously, Π depends upon the initial conditions $h(t_0, x, y,)$ and the interval $\tau = t_1 - t_0$. We apply Bellman's principle of optimality [*Bellman,* 1957], which states that if J is required to be a minimum during an interval $\tau = t_1 - t_0$, then it is necessary that J evaluated over a shorter interval $(\tau - \Delta)$ be a minimum also. To apply this principle, the integral on the right-hand side of equation 8 is written as the sum of two integrals, one evaluated over a short time interval $(t_0, t_0 + \Delta)$ and the other evaluated over the rest of the interval, i.e., $(t_0 + \Delta, t_1)$

$$\Pi[h(t_0, x, y), t_1 - t_0]$$
$$= \min_{\substack{T(x,y,t)\\ t\epsilon[t_0,t_0+\Delta]}} \int_{t_0}^{t_0+\Delta} \int_R (h - \hat{h})^T(h - \hat{h})\, dR\, dt$$
$$+ \min_{\substack{T(x,y,t)\\ t\epsilon[t_0+\Delta,t_1]}} \int_{t_0+\Delta}^{t_1} \int_R (h - \hat{h})^T(h - \hat{h})\, dR\, dt \quad (11)$$

For small Δ, the first double integral on the right-hand side of equation 11 can be written as

$$\int_{t_0}^{t_0+\Delta} \int_R (h - \hat{h})^T(h - \hat{h})\, dR\, dt$$
$$= \Delta\cdot \int_R (h - \hat{h})^T(h - \hat{h})\, dR \quad (12)$$

By virtue of the definition in Equation 10, the second term on the right-hand side of Equation 11 is recognized as the functional $\Pi\ [h(t_0 + \Delta, x,y), \tau{-}\Delta]$. Just as a function can be expanded about a point in Taylor series, this functional can be expanded about the point $(h(t_0,x,y), \tau)$ by a series as follows:

$$\Pi[h(t_0 + \Delta, x, y), \tau - \Delta]$$
$$= \Pi[h(t_0, x, y), \tau]$$
$$+ \Delta \int_R \left(\frac{\delta\Pi}{\delta h}\right)^T \bigg|_{t=t_0} \frac{\partial h}{\partial t}\cdot dR$$
$$+ \frac{\partial \Pi}{\partial \tau}\bigg|_{t=t_0} \cdot(-\Delta) + \cdots \quad (13)$$

where $\delta(\cdot)|\delta(\cdot)^T|_{t=t_0}$ is a *functional* partial derivative [*Goldstein,* 1950] evaluated at $t = t_0$, and the superscript T indicates a transpose. Substituting equation 12 and equation 13 into equation 11 and dropping higher-order terms in Δ

$$\Pi[h(t_0, x, y), \tau]$$
$$= \min_{\substack{T(x,y,t)\\ t\epsilon[t_0,t_0+\Delta]}} \left\{\Delta\cdot \int_R (h - \hat{h})^T(h - \hat{h})\, dR\right.$$
$$+ \Pi[h(t_0, x, y), \tau]$$
$$+ \Delta\cdot \int_R \left(\frac{\delta\Pi}{\delta h}\right)^T \bigg|_{t=t_0} \frac{\partial h}{dt}\cdot dR$$
$$\left. - \Delta\cdot\frac{\partial \Pi}{\partial \tau}\bigg|_{t=t_0}\right\} \quad (14)$$

Because $\Pi\ [h(t_0, x, y), \tau]$ and

$$\Delta\cdot\{\partial\Pi\ [h(t_0, x, y), \tau]\}/\partial\tau$$

do not depend upon $T(x, y, t)$ defined over the interval $[t_0, t_0 + \Delta]$, they may be taken out of the minimization operation. Therefore, after canceling $\Pi\ [h(t_0, x, y), \tau]$ on both sides, equation 14 becomes

$$\Delta\cdot\frac{\partial\Pi[h(t_0, x, y), \tau]}{\partial\tau}$$
$$= \min_{\substack{T(x,y,t)\\ t\epsilon[t_0,t_0+\Delta]}} \left\{\Delta\cdot \int_R (h - \hat{h})^T(h - \hat{h})\right.$$
$$\left. + \left(\frac{\delta\Pi}{\delta h}\right)^T \bigg|_{t=t_0} \frac{\partial h}{\partial t}\, dR\right\} \quad (15)$$

In the limit as $\Delta \rightarrow 0$, $t \rightarrow t_0$, equation 15 reduces to

$$\frac{\partial\Pi[h(t_0, x, y), \tau]}{\partial\tau}$$
$$= \min_{\substack{T(x,y,t)\\ t\epsilon[t_0,t_0+\Delta]}} \left\{\int_R (h - \hat{h})^T(h - \hat{h})\right.$$
$$\left. + \left(\frac{\delta\Pi}{\delta h}\right)^T \bigg|_{t=t_0} \cdot\frac{\partial h}{\partial t}\, dR\right\} \quad (16)$$

Derivation of equation 16 is valid for *any* initial condition, and there is nothing special about t_0. Therefore, equation 16 can be written in the more general form

$$\frac{\partial\Pi[h(t, x, y)]}{\partial\tau} = \min_{\substack{T(x,y,t)\\ t\epsilon[t_0,t_1]}} \left\{\int_R \left[(h - \hat{h})^T(h - \hat{h}) + \left(\frac{\delta\Pi}{\delta h}\right)^T \Big|_{t=t_0} \cdot \frac{\partial h}{\partial t}\right] dR\right\} \tag{17}$$

in which τ is now defined as $\tau = t_1 - t$.

If the quantities P and Q are now defined by the $(n + 1)$ dimensional vectors

$$\begin{aligned} P &= \mathrm{col}\,\{p_1, p_2, \cdots, p_{n+1}\} \\ &= \mathrm{col}\left\{\frac{\delta\Pi}{\delta h}, 1\right\} \\ Q &= \mathrm{col}\,\{q_1, q_2, \cdots, q_{n+1}\} \\ &= \mathrm{col}\left\{\frac{\partial h}{\partial t}, (h - h)^T(h - h)\right\} \end{aligned} \tag{18}$$

then equation 17 can be written as

$$\frac{\partial\Pi}{\partial\tau} = \min_{\substack{T(x,y,t)\\ t\epsilon[t_0,t_1]}} \int P^T \cdot Q\, dR \triangleq \min_{\substack{T(x,y,t)\\ t\epsilon[t_0,t_1]}} \langle P, Q\rangle \tag{19}$$

$$= \min_{\substack{T(x,y,t)\\ t\epsilon[t_0,t_1]}} H[h, P, T, t] = H^0[h, P, t] \tag{20}$$

From equation 18 and equation 19 it is clear that the Hamiltonian $H = \langle P,Q\rangle$ can be evaluated if the adjoint variable $v(t,x,y)$, defined by

$$v(t, x, y) = \delta\Pi/\delta h \tag{21}$$

is known. This adjoint variable is the solution of the adjoint differential equation

$$L^*v = (\partial v/\partial t) - 2(h - \hat{h}) \tag{22}$$

with the *terminal* condition

$$(\partial v/\partial n)(t_1, x, y) = 0 \qquad x, y, \epsilon R \tag{23}$$

and the boundary condition

$$v(t, x', y') = 0 \qquad x', y' \epsilon\, \partial R \tag{24}$$

where the adjoint operator L^* is defined by

$$\langle Lh, v\rangle = \langle L^*v, h\rangle \tag{25}$$

According to this definition, the adjoint operator L^* of L defined in equation 6 becomes

$$L^* = \frac{\partial}{\partial x}\left(T(x, y)\frac{\partial}{\partial x}\sigma(x, y)\right) + \frac{\partial}{\partial y}\left(T(x, y)\frac{\partial}{\partial y}\sigma(x, y)\right) \tag{26}$$

Stated this way, the adjoint problem is a final value problem and is very difficult to solve even with the help of a computer. Many computer runs are required to find one solution whose trajectory passes through the required final value specified by $v(t_1,x,y)$ in equation 23. This difficulty can be obviated by converting the final value problem into an initial value problem, which is relatively easier to solve. A final value problem can be solved as an initial value problem by solving it backwards in time, i.e., by starting the solution process at the terminal time and marching backwards until the initial time.

Formally, introducing a new independent variable τ, defined as

$$\tau = t_1 - t \tag{27}$$

the adjoint equation becomes

$$L^*v(t_1 - \tau, x, y) = -\frac{\partial v(t_1 - \tau, x, y)}{\partial\tau} - 2(h - \hat{h}) \tag{28}$$

with the *initial* condition

$$v(t_1, x, y) = 0 \qquad x, y\, \epsilon\, R \tag{29}$$

and the boundary condition

$$(\partial v/\partial n)(t_1 - \tau, x', y') = 0 \qquad x', y'\, \epsilon\, \partial R \tag{30}$$

The Steepest Descent Algorithm

Now, the procedure to identify the parameters can be described by the following steps.

0. Superimpose a finite-difference grid on the field and designate the coordinates of the grid points by the pair (l,m). Let $T_{l,m} = T(l\Delta x, m\Delta y)$ etc., where Δx and Δy indicate the grid size, respectively, along the x and y directions. Let Δt designate the size of step along the time axis.

1. Pick a nominal set of parameter values

$$\begin{aligned} T_{l,m} &= T_{l,m}^{(0)} \\ S_{l,m} &= S_{l,m}^{(0)} \end{aligned} \tag{31}$$

for all (l, m) and an appropriate choice of ∂R.

2. Using the values of T and S picked above the appropriate initial and boundary conditions, solve equation 5, the equation describing the dynamics of the water table over a small interval $[t_j, t_{j+1}]$. This equation is solved in discrete space and discrete time using the hybrid computer method. (See Appendix).

3. Solve the adjoint equation given by equation 28 with the initial and boundary conditions specified respectively by equation 29 and equation 30. This equation is also solved by the hybrid computer method.

4. Evaluate the Hamiltonian by computing

$$H = \int_R \sum_{i=1}^{N+1} p_i q_i \, dR \tag{32}$$

in the digital portion of the hybrid computer.

5. Compute the functional gradient of the Hamiltonian H with respect to the parameter T

$$\text{i.e., } \delta H/\delta T = -(\text{grad } h)\cdot(\text{grad } v) \tag{33}$$

is evaluated digitally using difference approximations.

6. Update the values of T according to

$$T_{l,m}^{(i+1)} = T_{l,m}^{(i)} - k(\delta H/\delta T)_{l,m} \qquad k > 0 \tag{34}$$

7. Check convergence of the iterative scheme by using a suitable criterion. For instance, convergence may be assumed if

$$(T_{l,m}^{(i+1)} - T_{l,m}^{(i)})/T_{l,m} < \epsilon_{l,m}(T) \quad \text{for all } l,m \tag{35}$$

If this condition is not satisfied, replace $T_{l,m}^{(i)}$ by $T_{l,m}^{(i+1)}$ and go back to step 2; otherwise go to step 10. The heuristic element in the algorithm enters the program in the following way.

8. If step 7 has been performd, say, for five times and still no convergence has occurred, then there is a possibility that the initial choice of S or ∂R or both is not good. Then the algorithm procedure of the computer is manually interrupted and control transferred to step 9.

9. Change the values of S by changing suitable numbers in the computer memory, or change ∂R by adjusting some wires on the analog patch board, or perform both the adjustments. Transfer control to step 2.

10. If the criterion in step 7 is satisfied but the resulting values of T, S, and ∂R are not physically plausible from geological and hydrological considerations, transfer control to step 9. If the resulting values of T, S, and ∂R are acceptable, go to next step.

11. Repeat the procedure for each value of j until the entire period is covered. Print the values of $T_{l,m}$, $S_{l,m}$, for all (l,m).

COMPUTER IMPLEMENTATION

The Role of the Hybrid Computer

A classical difficulty in implementing the preceding algorithm lies in getting the numerical solution of the dynamic and adjoint PDEs. Numerical solution of any PDE using digital computers is inherently expensive and time-consuming. In the identification procedure described in the preceding section, this difficulty is doubled because of the need to solve an additional PDE, namely the adjoint equation.

Conventionally, in solving a PDE by digital computer methods one would approximate the space derivatives usually by central differences and the time derivatives either by forward (explicit) or backward (implicit) difference quotients. Use of explicit difference approximations is characterized by the so-called stability problems [*Richtmyer and Morton,* 1967], unless the time increments are made uneconomically small from the point of view of computing time. Implicit approximations demand a simultaneous solution of a large number of equations, which involves time-consuming matrix inversion of large sparse matrices.

It is here that the hybrid computer makes its principal contribution. The hybrid computer used in this study uses a passive resistance network composed of fixed carbon film resistors (see Appendix) as a simultaneous equation solver and acts as a matrix inversion subroutine in a digital computer loop. Because a network of passive resistors relaxes almost instantaneously to its steady state, the time taken to invert the coefficient matrix associated with the set of finite-difference equations is very small when compared with conventional digital methods. Therefore, one can use the unconditionally stable implicit methods in conjunction with a hybrid computer and get both computational stability and speed in a single operation.

Furthermore, the speed gained by the use of analog hardware compensates for the additional work involved in solving the two-point boundary value problem.

The network subroutine that appears on the analog patch board preserves the geometry of the field being simulated. The flexibility offered by this network is an advantage that cannot be matched by pure digital systems.

Data Handling

In practice, the observed values $\hat{h}(x, y, t)$ are not available as a continuous function of the spatial coordinates and time. Usually, $\hat{h}$ will be recorded at discrete instants of time and at discrete points in space. Furthermore, the points of observation are not arranged, in general, in a regular array, whereas numerical solution of the PDEs using a symmetric finite difference grid is efficient and convenient. This leads to the situation where the N points of the finite difference grid at which h_α; $\alpha = 1, 2, \cdots, N$, is computed fail to coincide with the M points in the field at which $\hat{h}_\beta$; $\beta = 1, 2, \cdots, M$ has been reordered. This situation requires an initial processing of raw data, as described below, before the preceding algorithm can be applied.

1. Starting from water table data obtained from field measurements at the M points, water table contours showing equal elevations of water table are drawn.

2. A suitable grid is superimposed on this contour map and the average water table elevation at the center of each cell of the grid is computed using the Thiess method. The second grid thus obtained by connecting the center points of the first grid is the one that is used to write difference approximations of the continuous partial derivatives.

3. Data relating to the net accretion to the water table, i.e., Q, are likewise treated. The net accretion to the water table is obtained by taking the algebraic sum of all inflows and outflows. Typically, major components of inflow are precipitation and artificial recharge, and the principal component of outflow is pumping for consumptive use.

At this stage, the data are ready for computer use. It is useful to remember that from now onwards 'observed values h at the finite difference grid points' imply the values obtained after the aforementioned operations on the raw data.

A base period of six years (1950–1951 to 1955–1956) has been chosen for this study. There is no special significance attached to this period, except that the results of an analog computer study [*Blevins,* 1965] during that period are available for comparison purposes.

RESULTS AND DISCUSSION

The San Fernando Valley Basin

The technique described in the earlier sections was applied to study the San Fernando Valley basin in the City of Los Angeles, and some typical results are reported. The San Fernando Valley is roughly elliptical in shape, with a major axis of about 24 miles and a minor axis of approximately 12 miles, and covers an area of about 120,000 acres of the valley floor. The valley is surrounded on almost all sides by mountains which are, in general, impermeable to the movement of water. The Los Angeles River system, extending along the southern edge of the valley floor and going out of the Los Angeles Narrows into the Los Angeles Coastal Plain, is the major drainage medium of the basin. The source of ground water is by percolation, surface runoff from valley slopes, and spreading of imported waters. Disposal of the supply, other than export, consumptive use, and runoff is by relatively small amounts of underflow out of the area at the Narrows.

The western part of the valley, generally composed of fill materials that have a high clay content, transmits water at a relatively slow rate and exhibits a specific yield of about 0.02. The eastern portion of the valley, generally composed of coarse deposits of gravel and sand, transmits water at a relatively greater rate and exhibits values of specific yield as high as 0.3 to 0.4. This gravel-sand complex constitutes about one-third of the valley floor area and contains about two-thirds of the ground-water storage capacity of the basin. The pattern of groundwater flow, in general, is in a southeasterly direction from the recharge areas in the alluvial cones along the edges of the valley fill towards the Narrows. Most of the pumping wells supplying water to the cities of Glendale and Burbank are located in this southeastern segment of the valley.

Computed Water Table Contours

Some of the typical water table elevations obtained from the computer model and those that were measured at the test wells are plotted as contour lines and shown in Figure 1 and Figure 2. In these figures, the solid lines correspond to the observed water table, and the dashed lines correspond to the computer response. In general, maximum errors and anomalies appear to occur in the west-central section of the valley, i.e., in the vicinity of Lankershim boulevard and Vanowen street. This anomaly is perhaps due to a 'dome-like feature' [*Corbato*, 1960] in the depth of the water-bearing formations in that area.

Computed Transmissibility Contours

Contours of equal transmissibility corresponding to the periods ending in 1953–1954 and 1955–1956 are shown in Figures 3 and 4, respectively. Interpretation of the significance of the transmissibility contours is more difficult, because they are defined as the product of the 'equivalent permeability at a point (x,y)' and the depth of water-bearing formations at that point of the aquifer. Therefore, a necessary condition for a meaningful interpretation of these contours is a knowledge of the depth of the saturated portion of the aquifer. If contours on the base of the valley fill are available, it is possible to deduce the thickness of the saturated portion. Designating the elevation of the base of the valley fill above a reference as $f(x,y)$, the thickness $b(x,y,t)$ of the saturated portion of the aquifer at any time t can be written as

$$b(x, y, t) = h(x, y, t) - f(x, y) \qquad (36)$$

Because $T(x,y,t)$ is defined as

$$T(x, y, t) = K[x, y, h(x, y, t)] \cdot b(x, y, t) \qquad (37)$$

equation 36 and equation 37 help to determine the 'equivalent permeability at the point (x,y) on the ground surface,' i.e., $K(x,y,h(x,y,t))$.

In the present case [*California State Water Rights Board*, 1961] the depth of water-bearing formations is, in general, small, decreases along

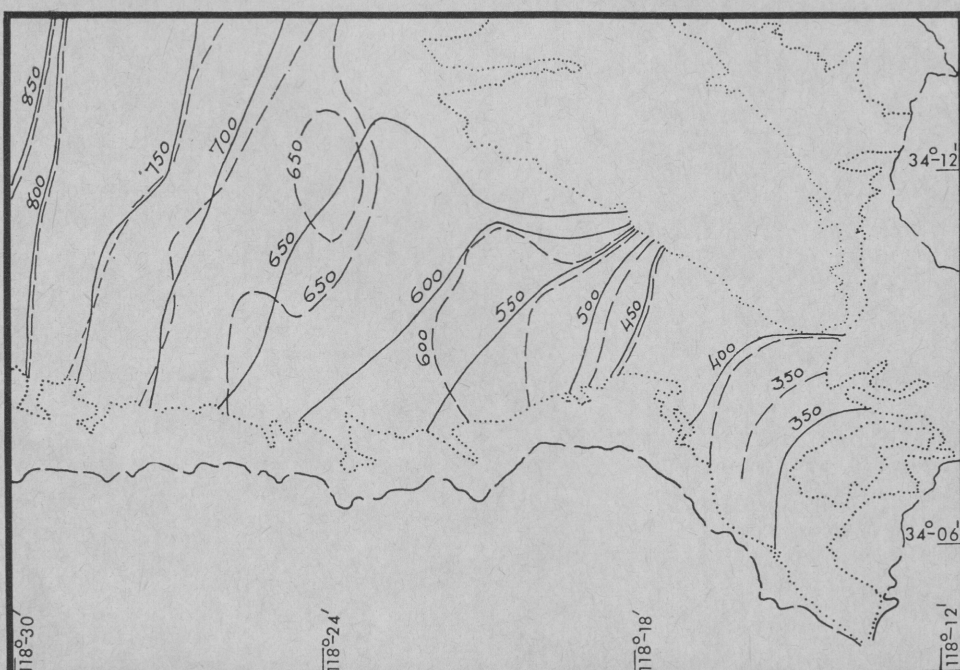

Fig. 1. Observed and computed water table contours at the end of 1953–54 in the eastern portion of the San Fernando Valley. The dotted lines indicate boundary of recent alluvium. The dashed lines correspond to observed water table contours, and the solid lines indicate response of the hybrid computer model. The numbers refer to the elevation in feet of water above a reference. The superimposed finite difference grid is not shown.

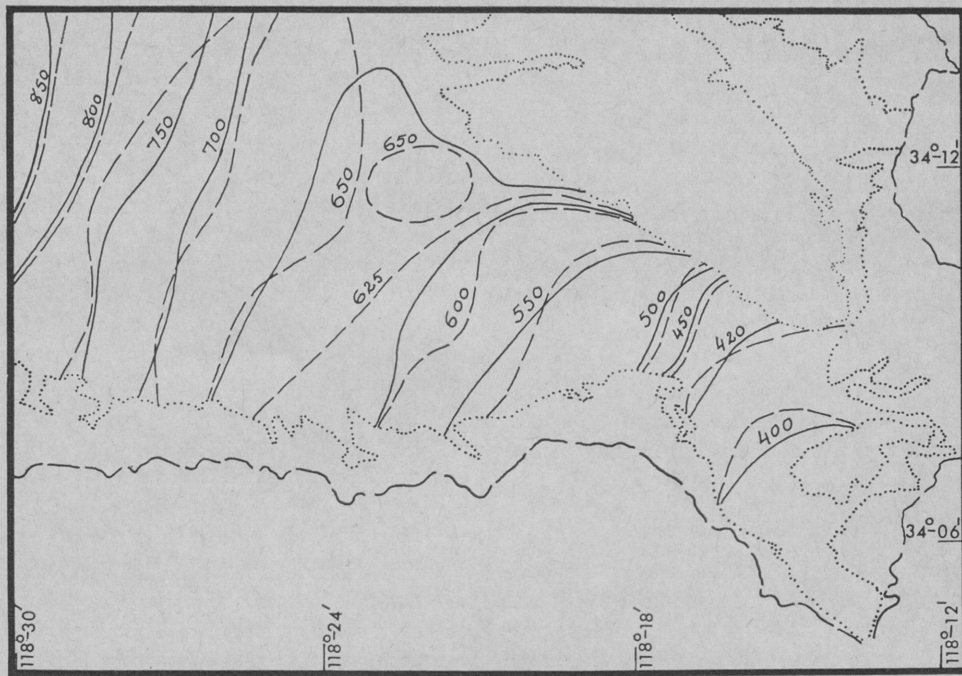

Fig. 2. Observed and computed water table contours at the end of the time period 1955–1956. The notation is same as the one used in Figure 1.

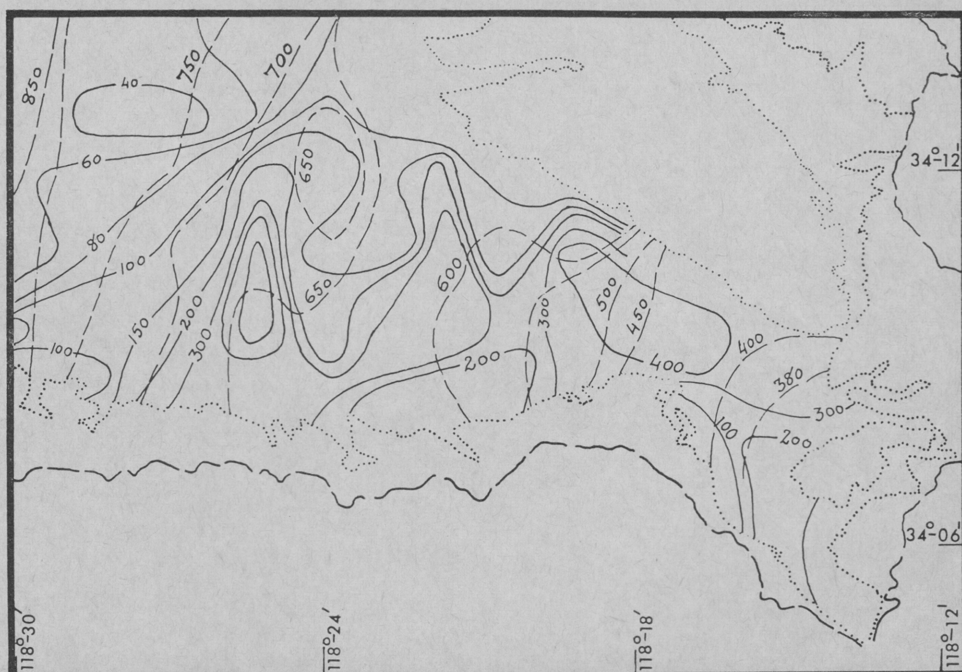

Fig. 3. Observed water table and computed transmissibility contours at the end of the time period 1953–1954. The solid lines are lines of equal transmissibility. The water table contours (dashed lines) are identical to those shown in Figure 1. This kind of plotting helps to deduce the dependence of T on h. The numbers on the solid lines indicate transmissibility in acre-ft/yr/ft.

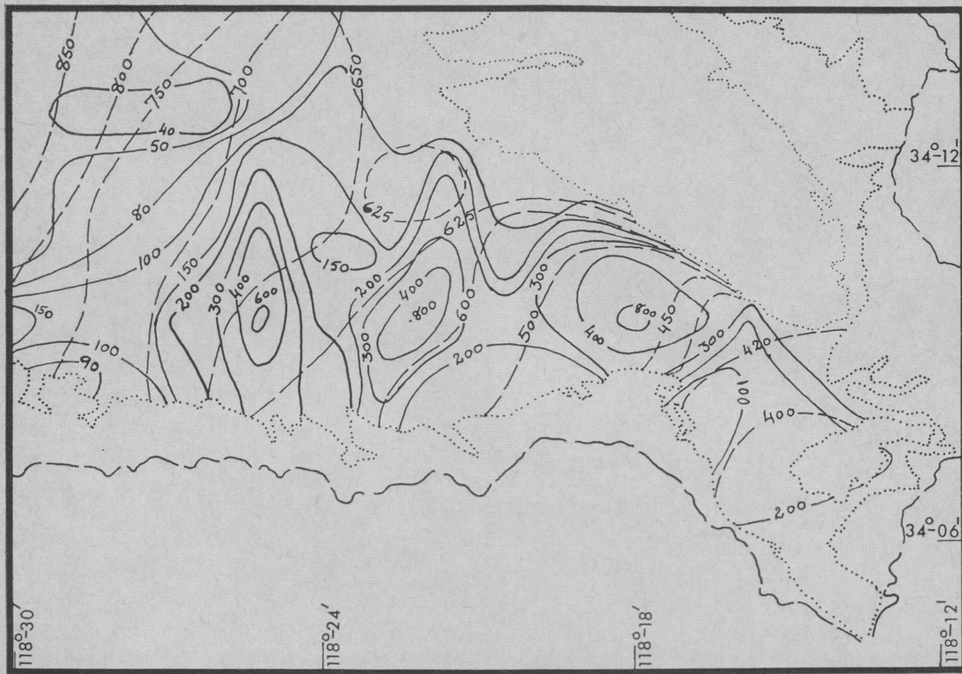

Fig. 4. Observed water table and computed transmissibility contours at the end of the time period 1955–1956. The notation is same as that used in Figure 1 and Figure 3.

the bed of the Los Angeles River, and becomes very small at the Narrows. Therefore, the transmissibilities along the river bed are consistently low, even though the bed offers high permeability. On the other hand, high transmissibilities in the central portions of the valley are due to large depths in water-bearing formations and are not necessarily indicative of high permeabilities.

There is a general tendency for the transmissibilities to decrease with declining water tables.

Computed Storage Coefficient Contours

Contours of equal storage coefficients are plotted in Figure 5 and Figure 6. The storage coefficients also show a general tendency to decrease with falling water tables. Also, storage coefficient values exhibited less sensitivity to the changing water tables.

Boundary of the Aquifer

By virtue of the assumption made in deriving the equation of ground water motion, namely that the aquifer is bounded on all sides by vertical imaginary impermeable barriers, the boundary ∂R is not a function of h. The shape of the boundary that gives the best model can be seen from the shape of the resistance network on the analog patch board.

Comments

This work [*Vemuri,* 1968] demonstrates the computational advantages in combining analog (parallel) and digital (serial) hardware, not only in terms of the gain in computational time but also in terms of the flexibility offered by the system. Moreover, hybridizing the hardware creates a unique opportunity to amalgamate in one system basically different kinds of computational techniques, namely, the blending of an algorithmic procedure with a heuristic approach. This conjecture has been proved by demonstrating the application of the functional maximum principle and the steepest descent algorithm to identify one of the unknown parameters in the conventional way. Identification of the other two parameters was done heuristically, by taking full advantage of the flexibility offered by the computer via the analog patch board.

Finally, this work demonstrates the utility

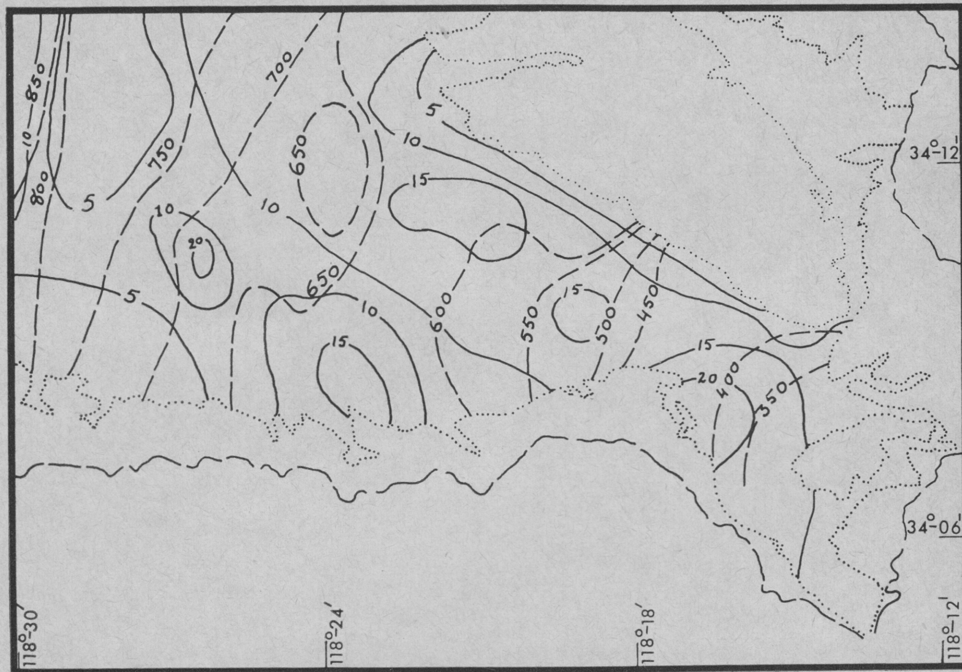

Fig. 5. Observed water table and computed storage coefficient contours at the end of the time period 1953–1954. The solid lines are lines of equal values of the storage coefficient, which is dimensionless.

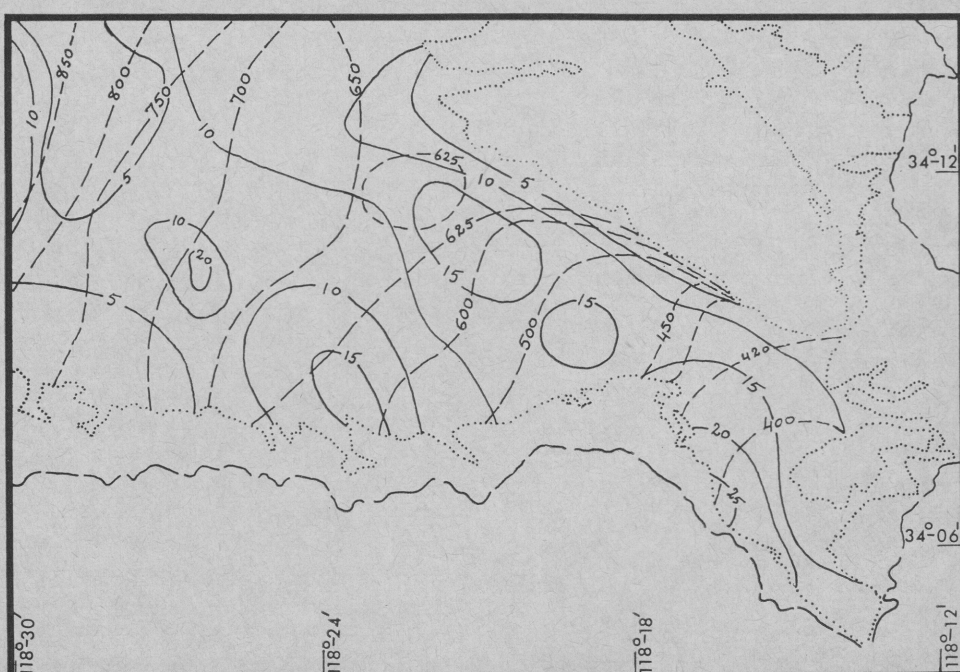

Fig. 6. Observed water table and computed storage coefficient contours at the end of the time period 1955–1956.

and power of a hybrid computer as a tool to practicing hydrologists. An ability to solve the inverse ground-water problem is not an end in itself; it is only a means to solve several other problems of related interest. For instance, this technique can be successfully employed to study problems related to sea water intrusion into fresh water aquifers and to ground water quality control investigations, to mention only two applications.

APPENDIX. HYBRID COMPUTER METHOD OF SOLVING EQUATION 1

A first step in solving equation 1 by the hybrid computer method is a definition of the finite-difference grid.

A 16 × 10 rectangular grid with square cells (of size $\Delta x = \Delta y = 6100$ feet) is superimposed on the valley floor. At each point of the grid, the continuous partial derivatives are replaced by suitably chosen finite-difference quotients. For this purpose, equation 1 is rearranged as

$$-\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = -\frac{S}{T}\cdot\frac{\partial h}{\partial t} + \frac{Q}{T} + \frac{1}{T}\cdot\frac{\partial T}{\partial x}\cdot\frac{\partial h}{\partial x} + \frac{1}{T}\cdot\frac{\partial T}{\partial y}\cdot\frac{\partial h}{\partial y} \qquad \text{(A-1)}$$

In Equation A-1, Q and h are in feet of water, S is dimensionless, and T is in cubic ft/yr/ft. However, it is customary to express T in acre-ft/yr/ft. For this purpose equation A-1 is rewritten as

$$-\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = -\frac{S}{43{,}560T}\frac{\partial h}{\partial t} + \frac{Q}{43{,}560T} + \frac{1}{T}\cdot\frac{\partial T}{\partial x}\cdot\frac{\partial h}{\partial x} + \frac{1}{T}\cdot\frac{\partial T}{\partial y}\cdot\frac{\partial h}{\partial y} \qquad \text{(A-2)}$$

with a scale factor, because 43,560 cubic feet correspond to one acre-foot. Replacing the space derivatives by central differences and the time derivative by a backward difference quotient and adding $\phi_{l,m,j+1}$ to both sides, equation A-2 becomes

$$-\phi_{l+1,m,j+1} - \phi_{l,m+1,j+1} + 5\phi_{l,m,j+1} - \phi_{l,m-1,j+1} - \phi_{l-1,m,j+1} = g_{l,m} \qquad \text{(A-3)}$$

where

$$\phi_{l,m,j} \triangleq h(l\Delta x,\ m\Delta y,\ j\Delta t)$$

and

$$g_{l,m} = \frac{-S_{l,m}(\Delta x)^2}{43{,}560\cdot T_{l,m}(\Delta t)}(\phi_{l,m,j+1} - \phi_{l,m,j}) + \frac{Q_{l,m}(\Delta x)^2}{43{,}560T_{l,m}} + \phi_{l,m,j+1}$$

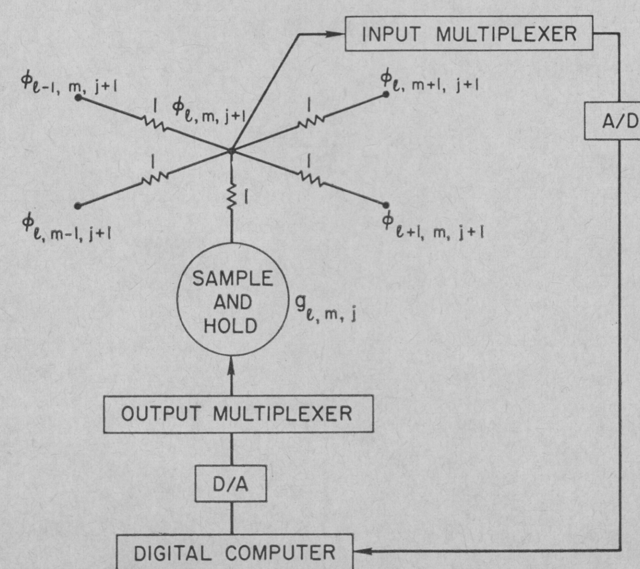

Fig. 7. Details of circuit at a typical node point of the resistance network.

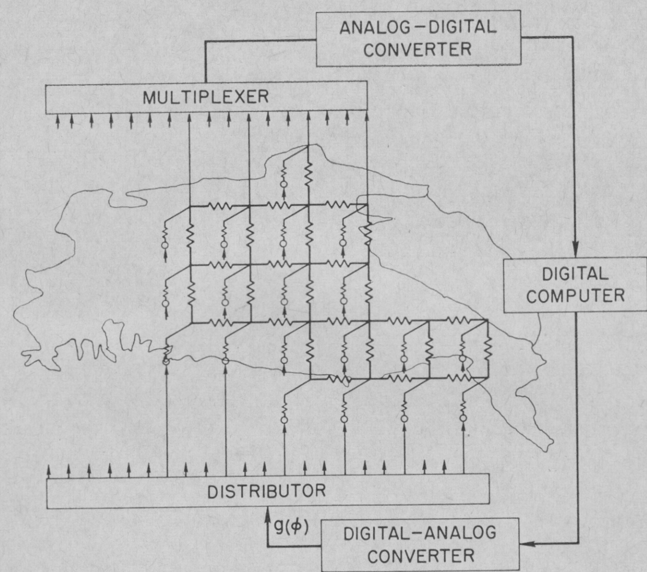

Fig. 8. Hybrid computer loop showing the configuration of circuit for the ground-water problem. The resistance network shown inside the valley area appears on an analog patch board.

$$+\frac{(T_{l+1,m} - T_{l-1,m})}{4T_{l,m}} \cdot (\phi_{l+1,m,j+1} - \phi_{l-1,m,j+1}) + \frac{(T_{l,m+1} - T_{l,m-1})}{4T_{l,m}} \cdot (\phi_{l,m+1,j+1} - \phi_{l,m-1,j+1}) \quad \text{(A-4)}$$

The quantity g is computed in the digital part of the hybrid computer.

The set of equations, of which equation A-3 is a typical member, is solved by using a pure passive resistance network analog. The network corresponding to a typical equation, such as equation A-3, is termed a node module and is shown in Figure 7. A schematic of the entire network simulating the eastern portion of San Fernando Valley is shown in Figure 8.

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