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On the Noninferior Index Approach to Large-Scale Multi-Criteria Systems

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ABSTRACT: *The noninferior index solution of multi-criteria optimization theory is studied as it relates to large-scale systems. The noninferior index set is related in a one-to-one manner to a family of auxiliary scalar index problems, where the auxiliary index is the inner product of the index vector J and a weighting vector α. Particular emphasis is placed on the functional relation between the noninferior index solutions and the weighting vector values. For a certain class of parameter optimization problems the entire noninferior index surface can be determined by solving the auxiliary index problem using only one value of the weighting vector α. The noninferior index solutions for such a parameter optimization problem are obtained for a large-scale system problem in water resource planning.*

I. Introduction

Many problems of a modern technological society are typically characterized by their complexity and variety. Problems of congestion, mounting cost and inefficiency of public services, and deterioration of the quality of our environment and its affect on various ecologies are just a few examples of the "blessings" of modern technological achievements. There is little question that the problem of characterizing and controlling large-scale civil, social or environmental systems so that they operate efficiently is one of the significant problems of our time.

While the agreement is nearly uniform in the recognition that large-scale problems exist and urgently need solution, convergence of opinion as to how to solve these problems is still lacking. It appears that there are three major ingredients to the study of a large-scale system, namely: the criterion function, the information available to effect meaningful control and finally the controller itself. Traditional control theory deals with one information set available to a single controller which attempts to extremize one criterion function.

Surely, one can easily visualize situations where there are more than one measure of performance. Typical examples of this category are vector-valued optimization or negotiation problems (**1, 2**) and nonzero sum differential games (**3, 4**). At a more practical level problems of public investment such as the operation of multipurpose river valley projects (**5**) and the design of transportation systems for urban areas are but two examples of large-scale systems requiring treatment with multiple objective functions.

The purpose of this paper is to investigate the optimization problem with multiple performance criteria and its relevance to large-scale systems of contemporary interest. Attention will be focused on one particular way of viewing performance *criteria* so as to select a "best" set. The emphasis here is on *criteria* because more than one criterion will be necessary in a general evaluation of the performance of large systems such as those already cited.

II. The Noninferior Index Problem

There already exist many references which treat the optimization of a scalar index for all types of system equations (**6–8**). The application of optimization principles to multi-criteria index problems is a topic which has not received such widespread attention. Yet the multi-criteria index problem is very important to the study of large-scale systems, because such systems generally require more than one measure of performance for an adequate system description. Also, the increasing use of systems analysis in non-technical fields such as advertising, education and urban development requires the formulation of subjective criteria which cannot be easily subsummed under a common scalar index such as cost or profit.

The general multi-criteria index problem will consist of scalar indexes $(J_1, \ldots, J_N)$ which may be considered as perfectly general system indexes for a system described by either linear or nonlinear algebraic or differential equations. These N scalar indexes will be considered "best" when they achieve the lowest values possible within the constraints of the problem, and, for the sake of definiteness, they will all be taken as positive quantities (i.e. $J_i > 0$; $i = 1, \ldots, N$). These are not stringent constraints on the problem since the indexes can be easily transformed to meet the above requirements. Also, the N scalar indexes will be considered as elements of an N-dimensional vector index J which belongs to the positive orthant† of Euclidean N-space E^N, designated by E^{N+}. This orthant is divided into admissible and inadmissible portions designated by Λ and Ω, respectively. A performance vector $J_0 \in E^{N+}$ belongs to Λ if and only if an admissible input exists which satisfies all constraints which might be imposed on the problem and which yields the performance vector J_0 when applied to the system. The region Ω is defined as the complement of Λ in E^{N+}. In the material which follows Λ is considered to be a simply connected region in E^{N+} which is continuous with respect to

† An orthant is the largest subspace of the vector space E^N within which the elements of the member vectors are of constant sign. The term orthant is a generalization of the terms quadrant and octant in two- and three-dimensional Euclidean space.

the domain of input controls or parameters (i.e. no index point in Λ is isolated from the remaining admissible points). Also, it is assumed that the origin of E^{N+} belongs to Ω since the trivial solution ($J_1^* = 0, \ldots, J_N^* = 0$) would be obtained otherwise. A schematic diagram of the two-dimensional index space E^{2+} is shown in Fig. 1.

The first major problem in dealing with the multi-criteria index problem is to define what is meant by a "best" vector index. When the system performance is described by a scalar index it is easy to see that the "best" or optimum index is one equal to the minimum (or maximum) value of the scalar performance index. A vector index, however, has direction as well as magnitude, and no standard mathematical comparison can be made in such a case. Therefore, consider the following definitions (**1–4**) which focus attention on a specific class of vector indexes to be sought.

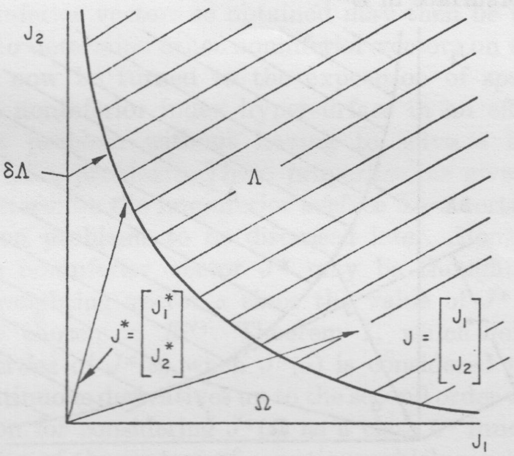

FIG. 1. Two-dimensional performance index space.

Definition 1: A vector index $J^* \in \Lambda$ is *optimal* with respect to Λ if $J_i^* \leqslant J_i$ $(i = 1, \ldots, N)$ for each $J \in \Lambda$.

Definition 2: A vector index $J^* \in \Lambda$ is *noninferior* with respect to Λ if there exists no other vector in Λ which is optimal with respect to J^*.

It can be seen immediately that optimality implies noninferiority, but the converse is not true unless $N = 1$ (i.e. the scalar index problem is considered). The optimal index problem although interesting theoretically (**2**) is not too important for practical engineering problems, because the optimum index solution may be found by optimizing any one of the scalar index elements without regard to the others. Thus, for such a problem $(N-1)$ of the index elements are redundant, because the control or input which minimizes one of the N elements minimizes the remaining $(N-1)$ elements as well. Therefore, the remainder of this paper will be devoted to the nonoptimal noninferior index problem.

III. Nonoptimal Noninferior Index Problem

Qualitatively, the noninferior index problem corresponds to a problem which has conflicting performance criteria, so that one scalar index element cannot be decreased in value along the noninferior hypersurface in E^{N+} without increasing the value of at least one other index element. The fact that noninferior vectors lie on the $(N-1)$ dimensional hypersurface which separates Λ and Ω can be shown (**9**). It is also known that the non-inferior vectors lie on the lower boundary of Λ, namely $\partial\Lambda$ (i.e. that portion of $\partial\Lambda$ which has the inward normal to Λ in the region E^{N+}). The portion of Fig. 1 labeled $\partial\Lambda$ would qualify as a noninferior surface in E^{2+}, and the noninferior index vectors which terminate on this boundary are designated by J^*. Thus instead of searching for a single "best" performance index vector the problem becomes one of defining a set of noninferior vectors which describe an $(N-1)$ dimensional hypersurface in E^{N+}.

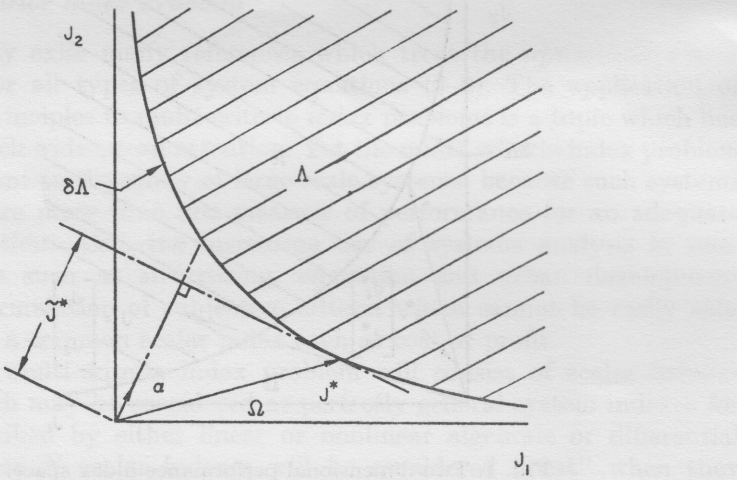

FIG. 2. Scalarized vector index problem.

Since the solution of scalar index optimal control problems is well documented for a large number of systems, the method of index scalarization offers an attractive means of solving for the noninferior index set. The method of scalarization consists of solving an auxiliary scalar index problem, where that auxiliary index is a scalar function of the vector performance index and a suitable weighting vector. It is well known (**3**, **4**), for example, that a sufficient condition for a performance vector J^* to be noninferior is that it must minimize $\tilde{J}$ for some value of α, where

$$\tilde{J} = \langle \alpha, J \rangle \triangleq \sum_{i=1}^{N} \alpha_i J_i; \quad \alpha, J \in E^{N+}, \tag{1}$$

$$\sum_{i=1}^{N} \alpha_i^2 = 1 \tag{2}$$

and

$$\tilde{J}^* = \min_{J \in \Lambda} [\tilde{J}] \triangleq \langle \alpha, J^* \rangle. \tag{3}$$

Here, the N dimensional vector α is the weighting vector, $\tilde{J}$ is the scalar auxiliary index which represents the orthogonal projection of J along the direction of the weighting vector α, and $\langle\,,\rangle$ indicates inner product. Figure 2 shows the geometric relations between $\tilde{J}$, α and J^* for the case $N = 2$. The generalization of these relations to higher dimensions is straightforward though not as easy to visualize.

As mentioned earlier, methods for solving scalar index optimization problems are well documented. The strategy is then to solve the optimization problem posed by Eq. (3) with suitable system constraints such as parameter constraints or control and state variable constraints if the system is described by a set of differential equations. This scalar index problem must be solved for all values of $\alpha \in E^{N+}$ in order to map out the corresponding noninferior surface. In practice, of course, the scalar problem will be solved for a finite number of weighting vector values which cover a given subregion of E^{N+}. The noninferior vectors so obtained may then be used in an interpolation process to determine other noninferior vectors on $\partial\Lambda$.

Attention will now be turned to the exposition of specific differential properties of the noninferior index hypersurface in an effort to solve the noninferior index problem without having to solve a large number of auxiliary scalar index problems. These properties, as given in Theorem I, will be used to determine the noninferior surface for a certain class of parameter optimization problems to be discussed later. Along the noninferior hypersurface the noninferior vector J^* may be thought of as a vector function of the weighting vector α since the value of J^* depends on the weighting vector chosen in E^{N+}. Theorem I, which follows, highlights differential properties of $J^*(\alpha)$ when $J^*(\alpha)$ is considered to be of class C^2 (i.e. $J^*(\alpha)$ has continuous derivatives up to the second order with respect to α).

The justification for considering $J^*(\alpha)$ as a class C^2 function is based on the differentiability of the system of equations which results as a necessary condition for the minimization of $\tilde{J}$. In an unconstrained parameter optimization problem, for example, the necessary condition for minimizing $\tilde{J}$ with respect to n independent variables $(x_1, \ldots, x_n)$ is that the first variation of $\tilde{J}$ with respect to each variable must be zero. The resulting system of n algebraic equations is then solved for the optimum values of $(x_1^*, \ldots, x_n^*)$ as functions of the weighting parameters $(\alpha_1, \ldots, \alpha_N)$. If the conditions of the implicit function theorem of calculus (**10**) are satisfied, the variables $(x_1^*, \ldots, x_n^*)$ are differentiable with respect to the parameters $(\alpha_1, \ldots, \alpha_N)$. The elements of the noninferior vector J^*, which are themselves differentiable functions of $(x_1, \ldots, x_n)$, are then differentiable with respect to $(\alpha_1, \ldots, \alpha_N)$ by the chain rule of differentiation. For an unconstrained optimum continuous control problem the differentiability of $J^*(\alpha)$ with respect to α can be shown by considering the differentiability of the Euler–LaGrange equations which result as necessary conditions for the minimization of $\tilde{J}$ (**9, 11**).

In Theorem I, the normalizing condition on α, given by Eq. (2), will be dropped so that the results of the theorem coincide with those presented later in connection with a particular class of unconstrained parameter optimization problems. This means that all N weighting coefficients are to be chosen independently from the domain of positive real numbers.

Theorem I. If $J^*(\alpha)$ is a solution to the problem

$$\min_{J\in\Lambda}[\tilde{J}] = \min_{J\in\Lambda}[\langle\hat{\alpha}, J\rangle]; \quad \hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_N) \in E^{N+},$$

then

(i) $$\sum_{i=1}^{N} \hat{\alpha}_i \frac{\partial J_i^*}{\partial \alpha}\bigg|_{\alpha=\hat{\alpha}} = 0,$$

(ii) $$\left[\sum_{i=1}^{N} \hat{\alpha}_i \frac{\partial^2 J_i^*}{\partial \alpha^2}\bigg|_{\alpha=\hat{\alpha}}\right]$$

is a positive semi-definite $(N\times N)$ dimensional matrix and

(iii) $$\frac{\mathrm{d}\tilde{J}^*}{\mathrm{d}\alpha_j}\bigg|_{\alpha=\hat{\alpha}} = J_j^*(\hat{\alpha}) \quad (j = 1, \ldots, N); \quad \tilde{J}^* = \min_{J\in\Lambda}[\tilde{J}].$$

Proof: Since J^* represents a minimum point of $\tilde{J} = \langle\hat{\alpha}, J\rangle$ for all $J \in \Lambda$ there is a hyperplane at J^* which separates Λ from the origin. This hyperplane has an inward normal to Λ proportional to the vector $\hat{\alpha}$. Since any admissible variation vector, $\mathrm{d}J$, about the point J^* must be directed toward Λ this means that the inner product of $\hat{\alpha}$ and $\mathrm{d}J$ must be greater than or equal to zero.

$$\sum_{i=1} \hat{\alpha}_i\, \mathrm{d}J_i \geqslant 0. \tag{4}$$

Thus must also hold true along the boundary of minimum points designated by $J^*(\alpha)$ for at least a small neighborhood of $J^*(\hat{\alpha})$. For a sufficiently small neighborhood of $\hat{\alpha}$, $\mathrm{d}\tilde{J}_i^*(\alpha)$ may be represented by

$$\mathrm{d}J_i^*(\alpha) \simeq \left[\frac{\partial J_i^*}{\partial \alpha}\bigg|_{\alpha=\hat{\alpha}}\right]\Delta\alpha + \frac{1}{2!}\Delta\alpha^{\mathrm{T}}\left[\frac{\partial^2 J_i^*}{\partial \alpha^2}\bigg|_{\alpha=\hat{\alpha}}\right]\Delta\alpha, \tag{5}$$

where $\mathrm{d}J_i^*(\alpha)$ denotes a variation along the noninferior surface, $\Delta\alpha^{\mathrm{T}}$ is the transpose of $\Delta\alpha = (\alpha - \hat{\alpha})$, and $i = 1, \ldots, N$.

After substituting these relations for $\mathrm{d}J_i^*$ $(i = 1, \ldots, N)$ into inequality (4) it is found that

$$\left[\sum_{i=1}^{N} \hat{\alpha}_i \frac{\partial J_i^*}{\partial \alpha}\bigg|_{\alpha=\hat{\alpha}}\right]\Delta\alpha + \frac{1}{2!}\Delta\alpha^{\mathrm{T}}\left[\sum_{i=1}^{N} \hat{\alpha}_i \frac{\partial^2 J_i^*}{\partial \alpha^2}\bigg|_{\alpha=\hat{\alpha}}\right]\Delta\alpha \geqslant 0 \tag{6}$$

for sufficiently small $\Delta\alpha$. If $\Delta\alpha$ is small enough so that the first term of inequality (6) dominates the left-hand side then

$$\left[\sum_{i=1}^{N} \hat{\alpha}_i \frac{\partial J_i^*}{\partial \alpha}\bigg|_{\alpha=\hat{\alpha}}\right] = 0 \tag{7}$$

in order to satisfy inequality (6) for $\Delta\alpha$ of arbitrary sign. This then leaves the second left-hand term in inequality (6) which must satisfy the inequality for all $\Delta\alpha$ sufficiently small. Therefore,

$$\left[\sum_{i=1}^{N} \hat{\alpha}_i \frac{\partial^2 J_i^*}{\partial \alpha^2}\bigg|_{\alpha=\hat{\alpha}}\right] \geqslant 0. \tag{8}$$

Finally, the differentiation of $\mathcal{J}^*$ with respect to α_j $(j = 1, \ldots, N)$ yields

$$\frac{d\mathcal{J}^*}{d\alpha_j} = \sum_{i=1}^{N}\left[\alpha_i \frac{\partial J_i^*}{\partial \alpha_j} + \delta_{ij} J_i^*\right] \quad (j = 1, \ldots, N), \tag{9}$$

where δ_{ij} is the Kronecker delta (i.e. $\delta_{ij} = 1$ when $i = j$; $\delta_{ij} = 0$ otherwise). Part (i) of this theorem indicates that at $\alpha = \hat{\alpha}$ the first N terms on the right sum to zero leaving the result

$$\left.\frac{d\mathcal{J}^*}{d\alpha_j}\right|_{\alpha=\hat{\alpha}} = J_j^*(\hat{\alpha}) \quad (j = 1, \ldots, N). \tag{10}$$

These conditions on $J_i^*(\alpha)$ hold true over any finite interval on which the continuity and differentiability assumptions are true.

The purpose of this section has been to define and exhibit one particular approach to the multi-criteria index optimization problem which should be of particular interest in the study of large-scale systems. The major advantage of the solution for a noninferior index surface is that it postpones the problem of selecting a desired performance vector until the performance character of the system is well understood. Also, the selection of a noninferior performance vector from the set of noninferior vectors assures the system analyst that no other admissible performance vector can offer better operating characteristics for all N index elements simultaneously. The next section will be devoted to a particular class of parameter optimization problems for which the results of the above theorem are especially useful in defining the noninferior index surface. For this particular problem class the entire noninferior surface can be defined by solving a scalarized problem of the form given by Eqs. (1) and (3) for only one value of the weighting vector α.

IV. An Application of the Noninferior Index Approach to Parameter Optimization

The class of parameter optimization problems to be studied in this section is related to and inspired by the posynomial function class introduced by Duffin *et al.* (**12**). The unconstrained parameter optimization problem involving a posynomial auxiliary index, $\mathcal{J}$, will be reviewed following a method first presented by Wilde and Beightler (8). An alternate solution, which utilizes the results of Theorem I, will then be presented and the class of parameter optimization problems will be expanded on the basis of the results of this theorem. The attractive feature of the class of parameter optimization problems to be studied here is that the noninferior index elements J_i^* $(i = 1, \ldots, N)$ can be related to the optimum auxiliary index $\mathcal{J}^*$ via a simple functional form involving the weighting coefficients α_i $(i = 1, \ldots, N)$. This property, when used in conjunction with Theorem I allows one to solve for $\mathcal{J}^*$ and J_i^* $(i = 1, \ldots, N)$ in terms of the weights α_i $(i = 1, \ldots, N)$. The use of such a property will become more apparent as the discussion proceeds.

Consider performance indexes of the form

$$J_j(x) = \prod_{i=1}^{n} (x_i)^{a_{ij}}, \quad x_i > 0 \quad (i = 1, \ldots, n; j = 1, \ldots, N), \tag{11}$$

where x represents an n-dimensional vector of positive independent variables, and a_{ij} are real numbers; either positive, negative or zero. The auxiliary index to be minimized is then given by the familiar linear sum of the J_i as

$$\bar{J} = \sum_{j=1}^{N} \alpha_j J_j(x); \quad \alpha_j \geqslant 0 \quad (j = 1, \ldots, N). \tag{12}$$

Since the auxiliary index, $\bar{J}$, is the sum of positive polynomials with positive coefficients the term *posynomial* was coined to describe the right side of the above Eq. (12) (**12**). A necessary condition for finding min $\bar{J}$ with respect to x is that (**8**)

$$\left(\frac{\partial \bar{J}}{\partial x_k}\right)^* = \sum_{j=1}^{N} \alpha_j a_{kj} (x_k^*)^{a_{kj}-1} \left[\prod_{i \neq k} (x_i^*)^{a_{ij}}\right] = 0, \tag{13}$$

$$\frac{1}{x_k^*} \sum_{j=1}^{N} a_{kj} \alpha_j J_j^*(x^*) = 0 \quad (k = 1, \ldots, n). \tag{14}$$

The asterisk notation in Eqs. (13) and (14) denotes a minimizing value for x. Now let performance weights w_j be defined by

$$w_j = \frac{\alpha_j J_j^*(x^*)}{\bar{J}^*(x^*)} \tag{15}$$

It is easy to see that

$$\sum_{j=1}^{N} w_j = 1. \tag{16}$$

Equations (14) provide n equations which when multiplied by $x_k^*/\bar{J}^*(x^*)$ $(k = 1, \ldots, n)$ yield the system of equations

$$0 = \frac{1}{\bar{J}^*(x^*)} \sum_{j=1}^{N} a_{kj} \alpha_j J_j^*(x^*); \; (k = 1, \ldots, n) \tag{17}$$

$$0 = \sum_{j=1}^{N} a_{kj} \frac{\alpha_j J_j^*(x^*)}{\bar{J}^*(x^*)} = \sum_{j=1}^{N} a_{kj} w_j; \; (k = 1, \ldots, n). \tag{18}$$

Thus, Eqs. (16) and (18) yield a system of $(n+1)$ linear equations for the N unknowns $w_j (j = 1, \ldots, N)$. The number $[N-(n+1)]$ is termed the degrees of freedom for the optimization problem (**12**). The present paper will deal with only those problems having zero degrees of freedom. For the particular case in which $N = (n+1)$, this system may be solved for unique values of $w_j (j = 1, \ldots, N)$ providing, of course, that the appropriate coefficient matrix is nonsingular. Note also that the values of $w_j (j = 1, \ldots, N)$ are constants independent of $\alpha_j (j = 1, \ldots, N)$. This is an important point which will be mentioned later.

Now, the minimum auxiliary index value $\bar{J}^*(x^*)$ may be written as

$$\bar{J}^*(x^*) = \prod_{j=1}^{N} [\bar{J}^*(x^*)]^{w_j} \tag{19}$$

since Eq. (16) is true. Therefore,

$$\tilde{J}^*(x^*) = \prod_{j=1}^{N} \left[\frac{\alpha_j J_j^*(x^*)}{w_j}\right]^{w_j}$$

$$\tilde{J}^*(x^*) = \prod_{j=1}^{N} \left(\frac{\alpha_j}{w_j}\right)^{w_j} \prod_{j=1}^{N} [J_j^*(x^*)]^{w_j} \tag{20}$$

However,

$$\prod_{j=1}^{N} [J_j^*(x^*)]^{w_j} = \prod_{j=1}^{N} \prod_{i=1}^{n} (x_i^*)^{a_{ij} w_j}$$

$$= \prod_{i=1}^{n} (x_i^*)^{\sum_{i=1}^{N} a_{ij} w}$$

$$\prod_{j=1}^{N} [J_j^*(x^*)]^{w_j} = \prod_{i=1}^{n} (x_i^*)^0 = 1. \tag{21}$$

Therefore,

$$\tilde{J}^*(x^*) = \prod_{j=1}^{N} \left(\frac{\alpha_j}{w_j}\right)^{w_j} \tag{22}$$

Equation (22) is a parametric representation of $\tilde{J}^*(x^*)$ in terms of the cost coefficients α_j $(j = 1, \ldots, N)$ and the weighting constants w_j $(j = 1, \ldots, N)$. Via Eq. (15) one can then determine the noninferior vector elements in terms of α_j, w_j $(j = 1, \ldots, N)$, and these elements are given by

$$J_j^*(x^*) = \left(\frac{w_j}{a_j}\right) \prod_{k=1}^{N} \left(\frac{\alpha_k}{w_k}\right)^{w_k} \quad (j = 1, \ldots, N). \tag{23}$$

Thus by solving the auxiliary minimization problem for one particular value of α_j $(j = 1, \ldots, N)$ the entire noninferior index surface is determined for all $\alpha_j > 0$ by Eq. (23). This is independent of one's ability to solve Eqs. (16) and (18) for w_j $(j = 1, \ldots, N)$, because the values of w_j $(j = 1, \ldots, N)$ are determined once the auxiliary index problem is solved for a particular set of α_j $(j = 1, \ldots, N)$. Although the above derivation is based on necessary conditions for the minimization of $\tilde{J}$ it can also be shown that the solution obtained is a minimum via sufficiency conditions which are presented elsewhere (**12**).

It is interesting to note that the result given in Eqs. (22) and (23) can be derived without using the results of Eqs. (19)–(22). The same relations for $\tilde{J}^*(x^*)$ and $J_j^*(x^*)$ $(j = 1, \ldots, N)$ in terms of α_j $(j = 1, \ldots, N)$ can be derived by using part (iii) of Theorem I and the fact that w_j $(j = 1, \ldots, N)$ in Eq. (15) are constant for all values of α_j $(j = 1, \ldots, N)$. From Eq. (15), one obtains

$$J_j^*(x^*) = \frac{w_j \tilde{J}^*(x^*)}{\alpha_j} \quad (j = 1, \ldots, N), \tag{24}$$

which when used in part (iii) of the theorem yields for any value of j

$$\frac{d\tilde{J}^*(x^*)}{d\alpha_j} = \frac{w_j}{\alpha_j} \tilde{J}^*(x^*) \quad (j = 1, \ldots, N). \tag{25}$$

The general solution to this differential equation is given by

$$\mathcal{J}^*(\alpha_1, \ldots, \alpha_N) = (\alpha_j)^{w_j}\{f_j(\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_N)\} \quad (j = 1, \ldots, N). \tag{26}$$

Since this equation must hold for all j the general form for $\mathcal{J}^*$ is

$$\mathcal{J}^*(\alpha_1, \ldots, \alpha_N) = C\prod_{j=1}^{N}(\alpha_j)^{w_j}; \quad C > 0 \tag{27}$$

and

$$J_j^*(\alpha_1, \ldots, \alpha_N) = C\left(\frac{w_j}{\alpha_j}\right)\prod_{k=1}^{N}(\alpha_k)^{w_k} \quad (j = 1, \ldots, N). \tag{28}$$

This is the same parametric form for the noninferior index elements as that given by Eq. (23), the only difference being that the coefficient C replaces the more explicit constant $\prod_{j=1}^{N}(1/w_j)^{w_j}$.

A noteworthy advantage of this alternate derivation is that the result of Eq. (28) or Eq. (23) depends only on the fact that w_j $(j = 1, \ldots, N)$ are constants, and this fact depends in turn on the ability to replace $\partial J_j^*(x^*)/x_k$ with $J_j^*(x^*)/x_k^*$ in Eqs. (13) and (14). Suppose that instead of being a polynomial term $J_j(x)$ is a general function of x such that

$$\frac{\partial J_j(x)}{\partial x_k} = y_{kj}(x_k)\, J_j(x) \quad (j = 1, \ldots, N;\ k = 1, \ldots, n). \tag{29}$$

The basic substitution property used in deriving Eq. (14) from Eq. (1)3 would then be preserved, and the resulting conclusions which lead to Eq. (28) would also be valid. The general form of $J_j(x)$ which satisfies the property required by Eq. (29) is

$$J_j(x) = C_j \exp\left\{\sum_{k=1}^{n}\int^{x_k} y_{kj}(\xi)\,\mathrm{d}\xi\right\}; \quad C_j > 0. \tag{30}$$

Note that when $y_{kj}(x_k) = a_{kj}/x_k$ and $C_j = 1$ then

$$\left.\begin{aligned}
\sum_{k=1}^{n}\int^{x_k}\frac{a_{kj}}{\xi}\,\mathrm{d}\xi &= \sum_{k=1}^{n} a_{kj}\ln x_k, \\
\sum_{k=1}^{n}\int^{x_k}\frac{a_{kj}}{\xi}\,\mathrm{d}\xi &= \sum_{k=1}^{n}\ln (x_k)^{a_{kj}}, \\
\sum_{k=1}^{n}\int^{x_k}\frac{a_{kj}}{\xi}\,\mathrm{d}\xi &= \ln\prod_{k=1}^{n}(x_k)^{a_{kj}},
\end{aligned}\right\} \tag{31}$$

and

$$\left.\begin{aligned}
J_j(x) &= \exp\left\{\ln\prod_{k=1}^{n}(x_k)^{a_{kj}}\right\}, \\
J_j(x) &= \prod_{k=1}^{n}(x_k)^{a_{kj}} \quad (j = 1, \ldots, N).
\end{aligned}\right\} \tag{32}$$

Thus the posynomial term defined by Eq. (11) is a special case of the more general functional form of Eq. (30). The existence of an auxiliary minimum and its uniqueness are not guaranteed for the more general index form given by Eq. (30), but the extension of the noninferior index problem to this index form is certainly worthy of investigation. The following example will serve to illustrate some of the principles presented above for a multi-index problem whose index elements are of the form given by Eq. (30).

V. A Multi-criteria Problem in Water Resource Planning

Consider the problem of determining the optimum storage capacity of a reservoir subject to a specified set of release rules (**13**). Also, assume that a dam of finite height impounds water in the reservoir and that water is required to be released for various purposes such as flood control, irrigation, industrial and urban use, and power generation. The reservoir may also be used for fish and wildlife enhancement, recreation, salinity and pollution control, mandatory releases to satisfy the riparian rights of downstream users, and so forth. The problem is essentially one of determining the storage capacity of the reservoir so as to maximize the net benefits accrued.

It is not always straightforward, nor is it desirable, to express the benefits in terms of net income, because the procedure for comparing the economical and recreational benefits of a large public project with the dangers of cultural and social dislocations is not very clear at the outset. Under these circumstances the concept of *utility*, as used by economists, is far more useful than scalar-valued performance criteria.

Treatment of this problem in its entirety is beyond the scope of this paper. To demonstrate the point and for several computational reasons this problem is simplified as follows. Let J_1 be an indicator of the capital cost of the project which depends on the total man hours x_1 devoted to building the dam and also on the mean radius x_2 of the lake impounded in some fashion. The height h of the proposed dam can be related to the variable x_1 by an equation of the form

$$h = [e^{x_1}(x_1)^2]^{1/a}; \quad a \text{ constant} > 0, \tag{33}$$

and the surface area of the reservoir A is

$$A = k_1 \Pi (x_2)^2; \quad k_1 \text{ constant} > 0. \tag{34}$$

Capital cost J_1 may be denoted by

$$\left.\begin{aligned} J_1 &= k_2 h^2 A; \quad k_2 \text{ constant} > 0, \\ J_1 &= k_1 k_2 \Pi [e^{2x_1/a}(x_1)^{4/a}(x_2)^2]. \end{aligned}\right\} \tag{35}$$

Similarly, let J_2 represent the water loss (volume/year) due to evaporation. This water loss is proportional to the surface area of the lake, so

$$\left.\begin{aligned} J_2 &= k_3 A; \quad k_3 \text{ constant} > 0, \\ J_2 &= k_1 k_3 \Pi (x_2)^2. \end{aligned}\right\} \tag{36}$$

The total volume capacity V of the reservoir is vital to the realization of the various goals set forth previously. This reservoir volume may be approximated by

$$\left.\begin{aligned} V &= hA, \\ V &= k_1 \Pi[e^{x_1/a}(x_1)^{2/a}(x_2)^2]. \end{aligned}\right\} \tag{37}$$

Since one is interested in formulating performance indexes to be minimized the third index J_3 will be taken as the reciprocal of V. The physical situation, in this case, leads to the qualitative assumption that J_1 and J_2 are quantities to be decreased, whereas V must be increased, in order to improve the system performance. Therefore, let

$$\left.\begin{aligned} J_3 &= \frac{1}{V}, \\ J_3 &= \frac{1}{k_1 \Pi}[e^{-x_1/a}(x_1)^{-2/a}(x_2)^{-2}]. \end{aligned}\right\} \tag{38}$$

The scalar indexes given by Eqs. (35), (36) and (38) represent the elements of the three-dimensional vector J, and the problem is to determine the noninferior index set J^* by the method described in the previous section.

In order to deal with a specific numerical form of the problem described above, the constants k_1, k_2, k_3 and a are chosen so that the three scalar performance indexes become

$$\left.\begin{aligned} J_1 &= e^{0\cdot01x_1}(x_1)^{0\cdot02}(x_2)^2, \\ J_2 &= \tfrac{1}{2}(x_2)^2, \\ J_3 &= e^{-0\cdot005x_1}(x_1)^{-0\cdot01}(x_2)^{-2}. \end{aligned}\right\} \tag{39}$$

The scalar auxiliary index $\bar{J}$ is then

$$\begin{aligned} \bar{J} = \alpha_1[e^{0\cdot01x_1}(x_1)^{0\cdot02}(x_2)^2] + \alpha_2[\tfrac{1}{2}(x_2)^2] \\ + \alpha_3[e^{-0\cdot005x_1}(x_1)^{-0\cdot01}(x_2)^{-2}], \end{aligned} \tag{40}$$

and the necessary conditions corresponding to Eq. (13) become

$$\left.\begin{aligned} 0 &= \left(\frac{\partial \bar{J}}{\partial x_1}\right)^* = \alpha_1\left[2J_1^*\left\{\frac{1}{200}+\frac{1}{100x_1^*}\right\}\right] + \alpha_3\left[-J_3^*\left\{\frac{1}{200}+\frac{1}{100x_1^*}\right\}\right], \\ 0 &= \left(\frac{\partial \bar{J}}{\partial x_2}\right)^* = \alpha_1\left[J_1^*\left\{\frac{2}{x_2^*}\right\}\right] + \alpha_2\left[J_2^*\left\{\frac{2}{x_2^*}\right\}\right] + \alpha_3\left[-J_3^*\left\{\frac{2}{x_2^*}\right\}\right], \end{aligned}\right\} \tag{41}$$

or

$$\left.\begin{aligned} 0 &= \alpha_1[2J_1^*] + 0 + \alpha_3[-J_3^*], \\ 0 &= \alpha_1[J_1^*] + \alpha_2[J_2^*] + \alpha_3[-J_3^*]. \end{aligned}\right\} \tag{42}$$

Now, let

$$w_1 = \frac{\alpha_1 J_1^*}{\bar{J}^*}; \quad w_2 = \frac{\alpha_2 J_2^*}{\bar{J}^*}; \quad w_3 = \frac{\alpha_3 J_3^*}{\bar{J}^*}. \tag{43}$$

Equations (42) and (43) when used with the unity condition of Eq. (16) lead to the following system of linear equations in w_1, w_2 and w,

$$\left.\begin{aligned} w_1+w_2+w_3 &= 1, \\ 2w_1+0-w_3 &= 0, \\ w_1+w_2-w_3 &= 0. \end{aligned}\right\} \tag{44}$$

The solution of Eqs. (44) can be found by inspection to be

$$w_1 = \tfrac{1}{4}; \quad w_2 = \tfrac{1}{4}; \quad w_3 = \tfrac{1}{2}. \tag{45}$$

Therefore, the noninferior surface for this problem in terms of the weighting coefficients α_1, α_2 and α_3 is found by combining the results of Eqs. (28) and (45). The noninferior index elements are

$$\left.\begin{aligned} J_1^*(\alpha_1,\alpha_2,\alpha_3) &= C(\tfrac{1}{4})\{(\alpha_1)^{-\frac{3}{4}}(\alpha_2)^{\frac{1}{4}}(\alpha_3)^{\frac{1}{2}}\}, \\ J_2^*(\alpha_1,\alpha_2,\alpha_3) &= C(\tfrac{1}{4})\{(\alpha_1)^{\frac{1}{4}}(\alpha_2)^{-\frac{3}{4}}(\alpha_3)^{\frac{1}{2}}\}, \\ J_3^*(\alpha_1,\alpha_2,\alpha_3) &= C(\tfrac{1}{2})\{(\alpha_1)^{\frac{1}{4}}(\alpha_2)^{\frac{1}{4}}(\alpha_3)^{-\frac{1}{2}}\}. \end{aligned}\right\} \tag{46}$$

The constant C is evaluated by solving Eqs. (42) for one particular value of the weighting vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and using any one of the resulting noninferior element values in Eqs. (46). This sample problem can be solved by using the Newton–Raphson numerical iteration technique (8) for minimizing $\tilde{J}$ at a given value of α. After a solution is obtained for the given value of α, the constant C is evaluated via Eqs. (46). Additional points on the noninferior surface can then be calculated via the parametric noninferior surface equations represented by Eqs. (46).

TABLE I

Sample Problem Results ($C = 2{\cdot}38$)

Data set	α_1	α_2	α_3	J_1^*	J_2^*	$1/J_3^*$	x_1^*	x_2^*
1	10^0	10^1	10^6	$1{\cdot}06\times10^3$	$1{\cdot}06\times10^2$	$4{\cdot}73\times10^2$	$1{\cdot}51\times10^2$	$1{\cdot}45\times10^1$
2	10^0	10^1	10^5	$3{\cdot}34\times10^2$	$3{\cdot}34\times10^1$	$1{\cdot}50\times10^2$	$1{\cdot}51\times10^2$	$8{\cdot}18\times10^0$
3	10^0	10^1	10^7	$3{\cdot}34\times10^3$	$3{\cdot}34\times10^2$	$1{\cdot}50\times10^3$	$1{\cdot}51\times10^2$	$2{\cdot}59\times10^1$
4	10^0	10^1	10^8	$1{\cdot}06\times10^4$	$1{\cdot}06\times10^3$	$4{\cdot}73\times10^3$	$1{\cdot}51\times10^2$	$4{\cdot}60\times10^1$
5	10^0	5×10^1	10^6	$1{\cdot}58\times10^3$	$3{\cdot}16\times10^1$	$3{\cdot}16\times10^2$	$3{\cdot}10\times10^2$	$7{\cdot}95\times10^0$
6	10^0	10^2	10^6	$1{\cdot}88\times10^3$	$1{\cdot}88\times10^3$	$2{\cdot}66\times10^2$	$3{\cdot}79\times10^2$	$6{\cdot}13\times10^0$
7	10^0	10^3	10^6	$3{\cdot}34\times10^3$	$3{\cdot}34\times10^0$	$1{\cdot}50\times10^2$	$6{\cdot}09\times10^2$	$2{\cdot}59\times10^0$
8	10^{-1}	10^1	10^6	$5{\cdot}95\times10^3$	$5{\cdot}95\times10^1$	$8{\cdot}41\times10^2$	$3{\cdot}79\times10^2$	$1{\cdot}09\times10^1$
9	5×10^{-1}	10^1	10^6	$1{\cdot}78\times10^3$	$8{\cdot}89\times10^1$	$5{\cdot}62\times10^2$	$2{\cdot}19\times10^2$	$1{\cdot}33\times10^1$
10	10^{-2}	10^1	10^6	$3{\cdot}34\times10^4$	$3{\cdot}34\times10^1$	$1{\cdot}50\times10^3$	$6{\cdot}09\times10^2$	$8{\cdot}18\times10^0$

These calculations were carried out for the weighting coefficient values shown in Table I, and the values of J_1^*, J_2^* and J_3^* were checked for each value of α by using the Newton–Raphson technique as well as the analytical solution given by Eqs. (46). For each value of α the resulting values of J_1^*, J_2^* and J_3^* calculated via Eqs. (46) agreed to eight decimal places with those given by the Newton–Raphson solution. In all cases the numerical iteration was carried out until convergence of $\bar{J}$ was satisfied to the eighth decimal place. The noninferior values of x_1 and x_2, which are denoted by x_1^* and x_2^* in Table I, were calculated by the Newton–Raphson technique since these values are calculated explicitly at the last step of the iteration process.

Note that $1/J_3^*$ is displayed in Table I rather than J_3^* since the reciprocal of J_3 represents the reservoir volume and thus has more physical significance than the quantity J_3. The numbers given in Table I, which have been rounded to two decimal places, may be thought of as units of money for J_1^*, water volume evaporated per year for J_2^* and total volume for $1/J_3^*$. One unit of J_1 may represent a dollar or a thousand dollars since the constants k_1, k_2 and k_3 have been chosen for numerical convenience rather than their physical significance. Thus the results of Table I are intended to convey only a qualitative feel for the performance trade-offs possible. A more detailed study of a specific problem would, of course, yield a quantitative significance for the numerical results obtained.

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References

(1) L. A. Zadeh, "Optimality and non-scalar-valued performance criteria", *I.E.E.E. Trans. Aut. Control*, Vol. AC-8, pp. 59–60, Jan., 1963.

(2) N. O. Da Cunha and E. Polak, "Constrained minimization under vector-valued criteria in linear topological spaces", in "Mathematical Theory of Control", edited by A. V. Balakrishnan and L. W. Neustadt, pp. 96–108, New York, Academic Press, 1967.

(3) A. W. Starr and Y. C. Ho, "Nonzero-sum differential games", *J. Optimization Theory Applic.*, Vol. 3, No. 3, pp. 184–206, March 1969.

(4) A. W. Starr and Y. C. Ho, "Further properties of nonzero-sum differential games", *J. Optimization Theory Applic.*, Vol. 3, No. 4, pp. 207–219, April 1969.

(5) D. C. Major, "Benefit–cost ratios for projects in multiple-objective investment programs", *Water Resources Res.*, Vol. 5, No. 6, pp. 1174–1178, Dec. 1969.

(6) M. Athans and P. L. Falb, "Optimal Control", New York, McGraw-Hill, 1966.

(7) S. J. Citron, "Elements of Optimal Control", New York, Holt, Rinehart and Winston, 1969.

(8) D. J. Wilde and C. S. Beightler, "Foundations of Optimization", Englewood Cliffs, New Jersey, Prentice-Hall, 1967.

(9) R. W. Reid and S. J. Citron, "On methods of calculating noninferior performance index vectors", A. A. & E. S. Report No. 70-1, Purdue University, Lafayette, Indiana, Jan. 1970.

(10) A. Taylor, "Advanced calculus", Boston, Ginn and Company, 1955.

(11) M. R. Hestenes, "Calculus of Variations and Optimal Control theory", New York, John Wiley, 1966.

(12) R. J. Duffin, E. L. Peterson and C. Zener, "Geometric Programming", New York, John Wiley, 1967.

(13) P. A. P. Moran, "A probability theory of a dam with continuous release", *Quart. J. Mathematics*, Oxford, Second Series, Vol. 7, No. 26, pp. 130–137, June 1956.