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Research note

On a property of a class of resistance networks with a ladder structure

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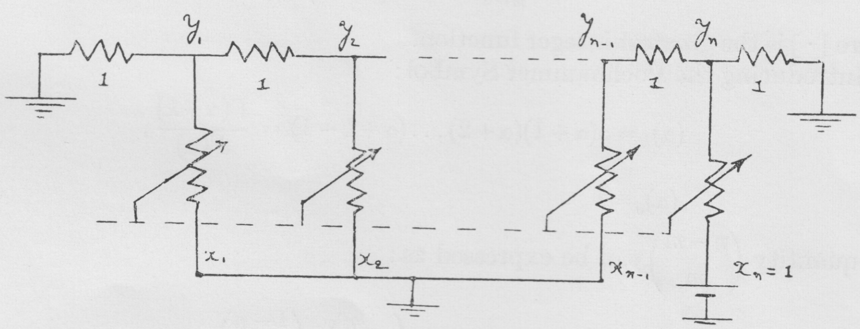
Kirchhoff's current law equations of a one-dimensional network of passive resistors yield a coefficient matrix which has a tridiagonal structure. If the magnitudes of the 'vertical' resistors in the ladder satisfy the relation in (1), then the entries of the inverse of the resulting Steltjes matrix can be expressed as Gegenbauer polynomials. Thus, the network could be used as a generator of these polynomials.

1. Validation

Consider a one-dimensional resistance network (see figure) in which the magnitude of the 'vertical' resistors, is a function of a variable $\eta > 0$ and let:

$$a = 3 + \eta. \tag{1}$$



Network configuration.

For a given set of input voltages x_i, an expression for the output voltages y_i can be written in vector matrix notation as:

$$Y = A_n^{-1} X, \tag{2}$$

where A_n is the $(n \times n)$ tridiagonal matrix:

$$A_n = \begin{bmatrix} a & -1 & & & & \\ -1 & a & -1 & & & \\ & -1 & a & -1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & -1 \\ & & & & & -1 & a \end{bmatrix} \tag{3}$$

and therefore A_n^{-1} can be written as:

$$A_n^{-1} = \frac{1}{D_n}\begin{bmatrix} D_{n-1} & D_{n-2} & \cdots\cdots & D_2 & D_1 & D_0 \\ D_{n-2} & D_1 D_{n-2} & \cdots\cdots & D_1 D_2 & D_1 D_1 & D_1 \\ D_{n-3} & D_1 D_{n-3} & \cdots\cdots & D_2 D_2 & D_1 D_2 & D_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ D_1 & D_1 D_1 & & D_1 D_{n-3} & D_1 D_{n-2} & D_{n-2} \\ D_0 & D_1 & & D_{n-3} & D_{n-2} & D_{n-1} \end{bmatrix}, \tag{4}$$

Where $D_n = D_n(a)$ is the determinant of the matrix A_n and is given by (Todd 1950):

$$\left.\begin{array}{ll} D_0(a)=1, & D_2(a)=a^2-1, \\ D_1(a)=a, & D_3(a)=a^3-2a. \end{array}\right\} \tag{5}$$

In general:

$$D_n(a) = a\,D_{n-1}(a) - D_{n-2}(a). \tag{6}$$

From (5) and (6) it is easy to recognize that D_{2n} is always an even polynomial in a and D_{2n+1} an odd polynomial. Therefore, a general expression for $D_n(a)$ can be written as:

$$D_n(a) = \sum_{m=0}^{[n/2]} (-1)^m \binom{n-m}{m} a^{n-2m}, \tag{7}$$

where $[\cdot]$ is the greatest integer function.

Introducing the Pochhammer Symbol:

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, \tag{8}$$

$$(\alpha)_0 = 1$$

the quantity $\binom{n-m}{m}$ can be expressed as:

$$\binom{n-m}{m} = \frac{(-1)^m\, 2^{2m}\left(-\frac{n}{2}\right)_m \left(\frac{1-n}{2}\right)_m}{(-n)_m\, m!}. \tag{9}$$

Substituting (9) in (7):

$$D_n(a) = a^n \sum_{m=0}^{[n/2]} \frac{\left(-\frac{n}{2}\right)_m \left(\frac{1-n}{2}\right)_m}{(-n)_m\, m!}\left(\frac{2}{a}\right)^{2m}. \tag{10}$$

Introducing a new variable $p = a/2$ and multiplying (10) by the quantity $\Gamma(1+n)/n!\,\Gamma(1) = 1$, D_n becomes:

$$D_n(a) = \frac{2^n\,\Gamma(1+n)}{n!\,\Gamma(1)} p^n \left\{ \sum_{m=0}^{[n/2]} \frac{\left(-\frac{n}{2}\right)_m \left(\frac{1-n}{2}\right)_m}{(-n)_m m!} . p^{-2m} \right\}. \tag{11}$$

The expression inside the braced parentheses of (11) is a polynomial approximation of the hypergeometric series. Using eqn. (15.4.1) of Abramowitz and Stegun (1964), namely:

$$ {}_2F_1(-\alpha,\beta;\gamma;x) = \sum_{m=0}^{\alpha} \frac{(-\alpha)_m(\beta)_m}{(\gamma)_m}\frac{x^m}{m!}, \tag{12}$$

where α is an integer and $\gamma=-m-l$; m, $l=0$, 1, 2, . . . , the right side of (11) becomes a valid expression for the Gegenbauer polynomial $C_n^{(1)}(p)=C_n^{(1)}(a/2)$ (Erdelyi 1953); that is:

$$D_n(a)=C_n^{1}(a/2). \tag{13}$$

The transition from (11) to (13) via (12) is valid for all integral values of n even or odd. For even n, α is equal to $(n/2)$ and for odd n, α is equal to $-(1-n)/2$.

Equations (2), (4) and (13) together give an expression for characterizing the output voltages y_i in terms of the input voltages x_i and the Gegenbauer polynomials.

If the input voltages x_i are all made to satisfy the condition:

$$x_i = \begin{cases} 0; & i=1,2,\dots,(n-1), \\ 1; & i=n, \end{cases} \tag{14}$$

then the output voltages y_i satisfy the relation:

$$y_1=C_0^{(1)}(a/2)/C_n^{(1)}(a/2)=1/C_n^{(1)}(a/2) \tag{15}$$

and

$$y_i=C_{i-1}^{(1)}(a/2)/C_n^{(1)}(a/2); \quad i=2,\dots,n. \tag{16}$$

2. Conclusion

The above analysis not only brings out an interesting property of a class of resistance networks but also points out a way of generating Gegenbauer polynomials using ladder networks. By varying all the 'vertical' resistors simultaneously using a ganged set of potentiometers, it seems possible to generate a class of Gegenbauer polynomials.

References

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