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Sensitivity Analysis Method of System Identification and Its Potential in Hydrologic Research

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Abstract. Sensitivity analysis should be an integral part of nearly every hydrologic study. At an elementary level, sensitivity analysis is useful in studying the relative sensitivity of the result to the data input. Studying the sensitivity of a hydrologic system to changes in its parameters and initial conditions makes it possible not only to gain insight into a system's behavior but to derive simple computational algorithms for the identification of unknown parameters. The fact that sensitivity analysis leads to simple initial-value problems makes it ideal for mechanization on an analog computer. The computational steps involved in implementing identification algorithms based on sensitivity analysis are relatively simpler than those based on such other methods as quasilinearization. The applicability of this method to identify both lumped and distributed hydrologic systems with deterministic or statistical input-output data is demonstrated.

INTRODUCTION

The general identification problem, in its wider sense, includes all techniques devised to determine and characterize a model numerically, either analytically or experimentally, from measurements performed on the signals entering and leaving the system and from possible additional knowledge of the structure and behavior of the system and the statistical variation of its parameters. In a restricted sense, identification is the process of finding the unknown characteristics, say the parameters, of a system from measured values of input-output as data.

Such an identification will clearly involve a method of taking measurements, a method of processing these measurements to bring them into a suitable format, and a decision, statistical or deterministic, to estimate the unknowns. It is implicit that the 'input' is not under the control of the observer. Basically, there are two different approaches to the process of 'determining' the unknowns. In one case, for example, the system is known a priori to be characterized by a differential equation of a given order, but the coefficients in the equation are unknown. In the second case, the problem is made nonparametric by making no assumptions either about linearity or about the nature of the dynamic equations.

The sensitivity analysis method of system identification belongs to the first category. The method is applicable to lumped as well as to distributed parameter models and also to deterministic or statistical input-output data. Indeed, sensitivity analysis should be an integral part of the study of any physical system, including hydrologic systems. For example, a knowledge about the relative sensitivity of a result to data input is very important in the analysis and design of hydrologic systems.

WHAT IS SENSITIVITY ANALYSIS

Sensitivity analysis, in a sense, implies a study of the sensitivity of a system's response due to disturbances. These disturbances may have a widely differing character: they may be small or large, momentary or permanent; they may be related to initial conditions or to coeffi-

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cients. Without delving too much into the mathematical niceties, the underlying theory of sensitivity analysis can be presented in the following way [*Tomovic*, 1962].

Consider a dynamic system represented by a differential equation

$$\dot{X} = F(X, t; \Pi); \qquad X(0) = X_0 \qquad (1)$$

where

$F = \{f_1, f_2 \cdots, f_n\}$, a vector function[2]. For instance, F may characterize an nth-order system of ordinary differential equations.
$X = \{x_1, x_2 \cdots, x_n\}$, the state variable vector of order n.
$\Pi = \{\pi_1, \pi_2, \cdots, \pi_m\}$, an m-dimensional parameter vector that includes the initial conditions as some of its components.

Since our interest is in the mathematical modeling of physical systems, it is necessary to stress some facts about (1). First of all, it is necessary to make a clear distinction between the vector-valued functions F and X on one hand and the vector Π on the other. The components of the vectors F and X are *functions* and so are represented by points in *function* (Hilbert) *space*, whereas the components of Π are represented by a point in the *finite-dimensional* parametric space. This implies that the parameter vector is a constant over a finite subinterval of the total observation interval $[t_0, t_1]$. This further implies that the case where the vector Π is time-varying, i.e., $\Pi = \Pi(t)$, is excluded from consideration here.

To facilitate the definition of a sensitivity coefficient, let a simplified version of (1) be

$$\dot{x} = f(x, t; \pi); \qquad x(0) = x_0 \qquad (2)$$

where f, x, and π are now scalar quantities. We are interested in determining the sensitivity of x, the solution of (2), to a perturbation in the value of the parameter π. Differentiating (2) with respect to π

$$\partial/\partial\pi(\partial x/\partial t) = (\partial f/\partial x)(\partial x/\partial \pi) + \partial f/\partial \pi \qquad (3)$$

Equation 3 is valid only if π is independent of t, and this is precisely the reason for excluding functions like $\pi = \pi(t)$ in the discussion of the preceding paragraph. If x is continuous and differentiable in both t and π, the order of differentiation in (3) can be interchanged to yield

$$\partial/\partial t(\partial x/\partial \pi) = (\partial f/\partial x)(\partial x/\partial \pi) + \partial f/\partial \pi \qquad (4)$$

In (4), the derivative $(\partial x/\partial \pi)$ has the following meaning:

$$\frac{\partial x(t, \pi)}{\partial \pi} = \lim_{\Delta \pi \to 0} \frac{x(t, \pi + \Delta\pi) - x(t, \pi)}{\Delta \pi}$$

and therefore gives a measure of the 'influence of a change in the parameter π on the solution x' and is defined as the *influence coefficient* or *sensitivity coefficient*. Depending upon the situation, this 'coefficient' could be a constant or a function.

In the present case, defining a sensitivity function $u(x, t, \pi)$ as

$$u = u(x, t, \pi) = \partial x(t, \pi)/\partial \pi \qquad (5)$$

equation 4 becomes

$$\dot{u} - (\partial f/\partial x)u = \partial f/\partial \pi; \quad u(0) = u_0 = 0 \quad (6)$$

Equation 6 is called the *sensitivity equation*. In general, if there are m parameters, as in (1), there will be m sensitivity equations whose solutions correspond to the m sensitivity functions. The sensitivity equation plays an important role in the study of dynamic systems, and it is useful to list some of its important properties [*Bihovski*, 1964]:

1. All the sensitivity equations are of the same order as the original equation.
2. The sensitivity equations are always linear, regardless of the linearity or nonlinearity of the original system.
3. If the original equation is also linear, then the structure of the sensitivity equation is identical to that of the former.
4. If the initial conditions of the original equation are independent of the parameter values, then the initial conditions of the sensitivity equation are zero.

From this it is clear that the sensitivity equation and its solution provide a good deal of insight into the dynamic behavior of a physical system. Furthermore, the linear character of the sensitivity equation facilitates its analytical treatment.

[2] In this section, unless otherwise stated, capital letters represent vectors and lower case letters stand for scalars or for the components of the corresponding vectors.

ROLE OF SENSITIVITY ANALYSIS IN PARAMETER IDENTIFICATION

A parameter identification problem, which is our main concern in this paper, may be formulated as follows. A dynamic system is described by (1) and the form of the equations, given by F, is assumed to be known. The goal is to construct a model, characterized by

$$\dot{Y} = F(Y, t; P); \qquad Y(0) = Y_0 \tag{7}$$

such that the model response Y and the dynamic response X of the actual system are close to each other in some acceptable sense. In (7), the vector

$$P = \{p_1, p_2, \cdots, p_m\} \tag{8}$$

represents the model parameters, as against Π representing the system parameters.

if the error vector $E(t, P)$ is defined as

$$E(t, P) \triangleq Y(t, P) - X(t) \tag{9}$$

then a convenient and acceptable measure to judge and achieve closeness of $Y(t)$ to $X(t)$ is to minimize

$$J(P) = J(E(t, P)) \triangleq \int_{t_0}^{t_1} E'E\, dt \tag{10}$$

In (10), the prime denotes a transpose. Therefore the argument under the integral sign is a scalar-valued function of time, and therefore J is a mere number. Essentially, the value of J is a function of the parameters, and therefore it can be minimized by adjusting the parameter values. If we can determine how J varies in the vicinity of P, then it becomes possible to find a *direction* in which to change P so as to reduce J in an optimum fashion.

If $J(P)$ is a smooth function of P, it can be expanded in Taylor series about a nominal P as

$$J(P + \Delta P) = J(P) + (\text{grad } J)' \Delta P + \text{higher order terms} \tag{11}$$

where (grad J) stands for the vector

$$(\text{grad } J) = \begin{Bmatrix} \partial J/\partial p_1 \\ \partial J/\partial p_2 \\ \vdots \\ \partial J/\partial p_m \end{Bmatrix} \tag{12}$$

and $\Delta P = (\Delta p_1, \cdots, \Delta p_m)'$ is a vector indicating the changes in parameter values; this is essentially what the computer has to evaluate. If (ΔP) is sufficiently small, the higher-order terms of (ΔP) in equation 11 can be neglected, leaving only linear terms. Once (11) is linearized in the said fashion, the choice of (ΔP) becomes less complicated.

It is instructive to note that the vectors (grad J) and ΔP are both of the same dimension and can be geometrically represented by two lines of different lengths and directions in the m-dimensional vector space. Therefore, a rotation and multiplication of one of the vectors by a positive scalar can make both of them identical. Mathematically, this process can be represented by [*Bellman*, 1962]

$$\Delta P = kT(\text{grad } J) \tag{13}$$

where k is the positive scalar, and T is an orthogonal matrix representing a rotation.

Substituting (13) in (11) and neglecting higher-order terms

$$J(P + \Delta P) = J(P) + k(\text{grad } J)' \cdot T \cdot (\text{grad } J) \tag{14}$$

i. e.

$$J(P + \Delta P) - J(P)/k = (\text{grad } J)'T(\text{grad } J)$$

or

$$(d/dk)J(P + \Delta P) = (\text{grad } J)'T(\text{grad } J) \tag{15}$$

From (15) it is clear that a change in P by ΔP produces a *decrease* of the value of J if and only if the right-hand side is negative, that is

$$(\text{grad } J)'T(\text{grad } J) < 0 \tag{16}$$

This is equivalent to saying that the transformation matrix T be negative definite [*Bellman*, 1962]. If T is any negative definite matrix and if it is envisaged to change the parameter vector according to (13), then the method is called the *gradient-descent method*. However, if T is chosen to be the negative of a unit matrix, then (13) becomes

$$\Delta P = -kI(\text{grad } J) \tag{17}$$

and then the method is called the *steepest-descent* method.

Computationally, therefore, the steepest-descent method attempts to modify the parameter vector by iteratively updating the vector used in the preceding step of the calculation. In other words, the value of P at the $(i + 1)$st iteration is given in terms of its value at the ith iteration as

$$P^{(i+1)} = P^{(i)} + m(\Delta P) \quad (18)$$

where the superscript i is the iterative index, and ΔP is given by (17). In this method, computation of (grad J) is perhaps the biggest computational task. In some cases a good deal of ingenuity is required to arrive at the best method of evaluating the gradient.

One way, for instance, is to compute the ith component of (17), after substituting (10) in (17) as

$$\Delta p_i = -k \sum_j \frac{\partial}{\partial p_i} \int_{t_0}^{t_1} e_j^2 \, dt \quad (19)$$

$$= -2k \sum_j \int_{t_0}^{t_1} e_j \frac{\partial e_j}{\partial p_i} \, dt \quad (20)$$

$$= -2k \sum_j \int_{t_0}^{t_1} e_j \frac{\partial y_j}{\partial p_i} \, dt \quad (21)$$

Equation 21 follows from (20) by virtue of the relation in (9). Invoking the definition of the sensitivity function, namely

$$u_i \triangleq (\partial y_j / \partial p_i) \quad (22)$$

equation 21 can be rewritten as

$$\Delta p_i = -2k \sum_j \int_{t_0}^{t_1} e_j u_i \, dt \quad (23)$$

Therefore, if the sensitivity function u is known, the gradient method described above can be implemented by computing Δp_i using equation 23, and the parameter values are updated according to the relation in (18). The sensitivity function u, in turn, is evaluated by solving the sensitivity equation.

IDENTIFICATION ALGORITHM

The computational process involved in the application of this method may be summarized as follows.

STEP 1: Assign a nominal value to the parameter vector P. This choice may be arbitrary and may be governed by physical considerations. Denote this vector by $P^{(i)}$ with $i = 0$.

STEP 2: With the current value of $P^{(i)}$ solve the system equations using a suitable computational algorithm.

STEP 3: Solve the associated sensitivity equations using appropriate initial and boundary conditions.

STEP 4: Using a suitable computational technique, compute the gradient of the criterion function J with respect to the parameters.

STEP 5: Update the parameter vector according to the relation in (18) until

$$|P^{(i+1)} - P^{(i)}| < Ep; \qquad Ep > 0 \quad (24)$$

STEP 6: If the above condition is not satisfied, replace i by $i + 1$ and go back to Step 2. If the above condition is satisfied, take the value $P^{(i+1)}$ as an estimate of the unknown vector Π.

The computational process described in the preceding steps may be represented pictorially in the form of a block diagram as shown in Figure 1 and is valid irrespective of the type of computer used.

APPLICATION IN HYDROLOGY

This computational process, termed the sensitivity analysis method of system identification, appears to hold great promise in hydrology research. The identification problem is not new in hydrology, and the literature is replete with examples of systems and their studies based on input-output data. Indeed, the scope and applicability of the method presented here can be widened by considering an identification problem as a branch of the so-called *inverse problems*. The other branch of the inverse problem is the more familiar synthesis or design problem. Even though the mathematical philosophy is the same, the goals of identification and design are different. The goal of design is to *construct* a physically realizable system that satisfies a specified type of input-output relation. In this section the utility of the sensitivity analysis method in the design and identification of systems will be considered by means of two examples.

The Design Problem (Example 1)

In the design of surge tanks, penstocks, navigation locks, and several other seemingly unrelated hydraulic systems, the following nonlinear ordinary differential equation occurs [*Reisman and Silvers,* 1967]:

$$d^2x/d\theta^2 + \psi(dx/d\theta)\ |dx/d\theta| + \omega^2 x = f(\theta) \qquad (25)$$

After the cessation of flow, the right-hand side becomes zero and then one is also interested in the decay of the transient to the steady state. From an engineering point of view, getting a solution is only a means but not an end in itself. The interest perhaps is to study the stability (inverse of sensitivity in a loose sense) of the system to be designed or to study the influence of the damping parameter ψ or the constant ω^2 on the response of the system.

After the system reaches steady state, (25) can be written as

$$\ddot{x} + \psi(\dot{x})^2\ sgn\ (\dot{x}) + \omega^2 x = 0 \qquad (26)$$

where the dot over x stands for differentiation with respect to the dimensionless time θ and

$$Sgn\ \dot{x} = \begin{cases} +1 & \text{if} \quad \dot{x} > 0 \\ -1 & \text{if} \quad \dot{x} < 0 \end{cases} \qquad (27)$$

Let the initial conditions on (26) be

$$x(t_0) = x_0 \qquad \dot{x}(t_0) = \dot{x}_0 \qquad (28)$$

and also let

$$|\ddot{x}| \leq c \qquad (29)$$

Equation 29 sets a limit on the acceleration of flow, which is a realistic physical constraint.

The coefficient of the second term in (26) has very important physical significance: ψ is the damping coefficient. It is very useful to study the sensitivity of the solution x to changes in ψ.

Differentiating (26) with respect to ψ and rearranging terms, the sensitivity equation becomes

$$\ddot{u} + 2\psi\dot{u}\dot{x} + \omega^2 u = -(\dot{x})^2\ sgn\ (\dot{x}) \qquad u(0) = 0 \qquad \dot{u}(0) = 0 \qquad (30)$$

Equations 26 and 30 can now be solved on an analog computer, and the solution of (30) is the required sensitivity function.

If, for instance, it is required to determine the 'best' damping coefficient to get a desired response, this sensitivity function is used in the computation of (grad J) and the algorithm of the preceding section implemented to 'identify' the 'best' ψ.

This example is chosen to illustrate another important point that is often ignored. It is known from the theory of differential equations that the solution x of (25) depends continuously on the

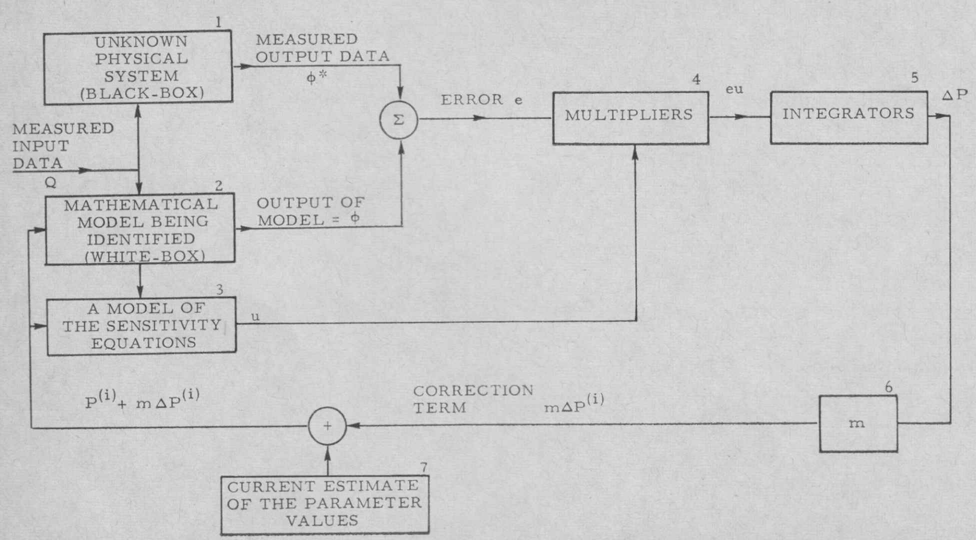

Fig. 1. A general block diagram showing the computational steps involved in the identification algorithm.

initial conditions and on the parameters. In this case, at the instant when $|\ddot{x}| \equiv c$, the solution does not depend analytically on the parameters. Designating this instant as $t = t_c$, it is seen that for $t > t_c$, equations 26 and 30 become

$$\psi \dot{x}^2 \, sgn\,(\dot{x}) + \omega^2 x = -c \qquad x(0) = x(t_c) \tag{31}$$

and

$$2\psi \dot{x}\dot{u} + \omega^2 x = -\dot{x}^2 \, sgn\,(\dot{x}) \qquad u(0) = u(t_c) \tag{32}$$

This situation calls for two different analog computer circuits, and the switch from one circuit to the other at the instant $t = t_c$ can be done automatically by means of amplitude comparators and switching circuits.

Identification of a System with a Random Input (Example 2)

In many problems in hydrology, the data collected historically, say $\phi^*(x, t)$, are not a deterministic quantity, not only because of the errors in the measurement but also because of an inherent randomness in the process itself. For instance, in modeling rainfall-runoff phenomena the rainfall and runoff are random processes, the rainfall being the input to and runoff the output from a watershed, regarded as a system here. Knowing the statistics [*Chow*, 1964] of the data on the rainfall Q and runoff ϕ^*, the task is to estimate the unknown values of the system parameters.

Let us assume that the input Q and the output ϕ of the model are related to each other via a partial differential equation, linear or nonlinear. For concreteness, let this model equation have the general structure

$$F(L\phi, Q, P) = 0 \tag{33}$$

where L is a row matrix of partial differential operators, Q represents the distributed rainfall as a function of time, and P is an m-dimensional parameter vector. Let us further assume that the measured input Q^* and output ϕ^* can be separated into a stationary signal part, Q_s and ϕ_s, and a random noise part, i. e.

$$\phi^* = \phi_s + \Delta\phi^* \qquad \phi = \phi_s + \Delta\phi$$
and
$$Q^* = Q_s + \Delta Q^* \qquad Q = Q_s + \Delta Q \tag{34}$$

and that their statistics are given respectively by the auto- and cross-correlation functions

$$R_{\phi^*\phi^*}(X, t, \tau) \triangleq E[\Delta\phi^*(X, t)\cdot\Delta\phi^*(X, t - \tau)] \tag{35}$$
$$R_{\phi^*Q^*}(X, t, \tau) \triangleq E[\Delta\phi^*(X, t)\cdot\Delta Q^*(X, t - \tau)]$$

With the above assumptions, the problem is to estimate the parameter vector P that minimizes the expected value of the criterion function. That is

$$\underset{P}{\text{Min}}\,\langle J\rangle = \underset{P}{\text{Min}}\left\langle \int_R \int_{t_0}^{t_1} (\Delta\phi^* - \Delta\phi)^T \cdot(\Delta\phi^* - \Delta\phi)\,dt\,dR \right\rangle \tag{36}$$

where $\langle x\rangle \equiv E[x]$ is the expected value of the random variable x.

The procedure to minimize J will be demonstrated now for the case of a scalar function ϕ, which greatly simplifies the algebra.

Using sensitivity analysis, it can be shown (see Appendix 1) that $\Delta\phi$ in this case is

$$\Delta\phi = \int_{t_0}^{t_1} \dot{u}(x, t - \tau)\cdot\Delta Q^*(x, \tau)\,d\tau \tag{37}$$

where $\dot{u}$ is the time derivative of the partial sensitivity function between $\Delta\phi$ and ΔQ. Inserting (37) in a simplified version of (36)

$$\langle J\rangle = \left\langle \int_0^R \int_{t_0}^{t_1} \left\{\Delta\phi^*(x, t) - \int_{t_0}^{t_1} \dot{u}(x, t - \tau) \cdot\Delta Q^*(x, \tau)\,d\tau\right\}^2 dt\,dR \right\rangle \tag{38}$$

The expected value of the gradient of J with respect to the parameters is now computed (see Appendix 2), and the ith component of $\langle \text{grad } J\rangle$ is found to be

$$\begin{aligned}\langle \text{grad } J\rangle_{p_i} &\triangleq \left\langle \frac{\delta J}{\delta p_i} \right\rangle \\ &= \int_R dR\left\{-2\int_{t_0}^{t_1} \frac{\partial \dot{u}(t - \tau)}{\partial p_i} R_{\phi Q}(t - \tau)\,d\tau \right. \\ &\quad + 2\int_{t_0}^{t_1} \frac{\partial \dot{u}(t - \tau)}{\partial p_i} \int_{t_0}^{t_1} \dot{u}(t - \tau) \\ &\quad \left. \cdot R_{\phi\phi}(t - \sigma)\,d\tau\,d\sigma\right\}\end{aligned} \tag{39}$$

Once the gradient is computed, the steps involved in the identification procedure are once again the same as the ones described earlier.

COMPUTER SIMULATION

So far the discussion has been deliberately kept independent of the nature of the computer used. From Figure 1 it is clear that one has to solve the system equations appearing in box 2 and the sensitivity equations appearing in box 3. Fortunately, both these equations are posed as initial value problems, and analog, digital, and hybrid computers are all suitable and efficient in solving initial value problems.

It is important to note the structural similarity between the system equations and the sensitivity equations, for this facilitates to a certain extent the programming of analog and digital computers. For instance, if a digital computer is used, the same numerical algorithm can be applied to solve both the equations. Some of the coefficients in both the equations are identical, which results in some savings in digital computer memory.

In studying systems characterized by ordinary differential equations, such as in example 1, the analog and hybrid computers lend themselves very efficiently to calculating the sensitivity functions. For instance, in example 1, the task of peak detection by amplitude comparators and the subsequent switching from one set of equations to another can be very effectively done by an analog computer. The task of computing m-sensitivity functions in the general case is another case in point. If P is an m-dimensional vector, the sensitivity equation must be solved repeatedly m times to get the m first-order sensitivity coefficients. This straightforward method is inefficient, because the homogeneous part of the sensitivity equation remains unchanged during each of these calculations, and only the independent term changes. There are elegant analog computer techniques to get all the sensitivity coefficients in a single computer run [*Tomovic,* 1963]. Furthermore, the steepest descent algorithm can be very conveniently implemented on the analog computer.

If the system under study is governed by a partial differential equation, then analog computers are not ideally suited to solve the equations, and digital computers are extremely time consuming. On such occasions, the hybrid computer [*Karplus,* 1964; *Vemuri and Dracup,* 1967] becomes very competitive with other computational tools. In any event, the choice of the computer used depends largely upon the particular problem under study and the computer available.

CONCLUSION

The sensitivity analysis of system identification is introduced, and an outline of implementing the algorithm on a computer is presented. The applicability of this method in hydrology research is discussed. The potential of this method in modeling rainfall-runoff relations of small watersheds is currently under investigation.

APPENDIX 1

Consider a linear, time-invariant, constant-coefficient system characterized by an ordinary differential equation with specified initial conditions. From elementary system theory it is well known that the impulse response of this system is the system-weighting function $W(t)$. Also, it is well known that the time derivative of the step response is identical to the impulse response. Therefore, if $G(t)$ is the step response of the given system, one can relate $G(t)$ to $W(t)$ by the relation

$$W(t) = \dot{G}(t) = dG(t)/dt \qquad \text{(A1)}$$

Recalling that the parameter vector P includes the initial conditions, we can legitimately talk about the sensitivity of the solution of the system differential equation to a change in the initial conditions. If the system is characterized by

$$\begin{aligned} \dot{x} &= f(x, t, P) \\ P &= \{p_0, p_1, \cdots, p_{m-1}\}; \quad p_0 = x(0) \end{aligned} \qquad \text{(A2)}$$

then $(\partial x/\partial p_0)$ is the sensitivity of the solution $x(t)$ to a step change in the initial conditions. That is

$$u(t, p_0) = \partial x/\partial p_0 \qquad \text{(A3)}$$

If the initial conditions, to start with, were to be zero, then $u(t, p_0)$ can be represented as the step response $G(t)$ of the system. Therefore

$$G(t) = u(t, p_0) \qquad \text{(A4)}$$

Combining (A1) and (A4)

$$W(t) = \dot{G}(t) = \dot{u}(t, p_0) \qquad \text{(A5)}$$

Thus, the time derivative of the sensitivity function (with respect to the initial conditions) is identical to the system weighting function. Using the convolution theorem, we can say that the response $\Delta\phi$ of the system to any excitation ΔQ can be written as

$$\Delta\phi = \int_{t_0}^{t_1} W(t - \tau) \cdot \Delta Q(\tau) \, d\tau \qquad \text{(A6)}$$

$$= \int_{t_0}^{t_1} \dot{u}(t - \tau) \, \Delta Q(\tau) \, d\tau \qquad \text{(A7)}$$

APPENDIX 2

COMPUTATION OF THE GRADIENT OF J

From equation 36

$$J = \int_R \int_{t_0}^{t_1} (\Delta\phi^* - \Delta\phi)^2 \, dt \, dR \qquad \text{(A2-1)}$$

By virtue of (37), equation A2-1 becomes

$$J = \int_R \int_{t_0}^{t_1} \left(\Delta\phi^* - \int_{t_0}^{t_1} \dot{u}(x, t - \tau) \cdot \Delta Q^*(x, \tau) \, d\tau \right)^2 dt \, dR \qquad \text{(A2-2)}$$

To simplify the notation, let

$$a = \Delta\phi^*$$
$$b = \dot{u}(x, t - \tau) \qquad \text{(A2-3)}$$
$$c = \Delta Q^*$$

Rewriting (A2-2) in the simplified notation

$$J = \int_R \int_{t_0}^{t_1} \left(a - \int_{t_0}^{t_1} b \cdot c \cdot d\tau \right)^2 dt \, dR \qquad \text{(A2-4)}$$

Because both a and c are derived from recorded data, any variation in J should come from a variation in b. If a variation δb of b causes a variation δJ in J, then

$$J + \delta J = \int_R \int_{t_0}^{t_1} \left(a - \int_{t_0}^{t_1} (b + \delta b) c \, d\tau \right)^2 dt \, dR \qquad \text{(A2-5)}$$

Subtracting (A2-4) from (A2-5) and simplifying the resulting expression

$$\delta J = \int_R \int_{t_0}^{t_1} \left\{ \int_{t_0}^{t_1} (\delta b)^2 c^2 \, d\tau + \int_{t_0}^{t_1} 2b(\delta b) c^2 \, d\tau - 2a \int_{t_0}^{t_1} (\delta b) c \, d\tau \right\} dt \, dR \qquad \text{(A2-6)}$$

Substituting the values of a, b, and c back in (A2-6), taking the expected values on both sides, and using the definitions of auto- and cross-correlation functions given in (35), equation A2-6 becomes

$$\langle \delta J \rangle = \int_R dR \int_{t_0}^{t_1} \delta\dot{u}(t - \tau) \cdot \int_{t_0}^{t_1} \delta\dot{u}(t - \tau) R_{\phi\phi}(\tau - \sigma) \, d\sigma \, d\tau + 2 \int_R dR \int_{t_0}^{t_1} \delta\dot{u}(t - \tau) \cdot \int_{t_0}^{t_1} \dot{u}(t - \tau) R_{\phi\phi}(\tau - \sigma) \, d\sigma \, d\tau - 2 \int_R dR \int_{t_0}^{t_1} \int_{t_0}^{t_1} \delta\dot{u}(t - \tau) \cdot R_{\phi Q}(\tau - \sigma) \, d\sigma \, d\tau \qquad \text{(A2-7)}$$

where $\langle x \rangle$ indicates expected value of the random variable x. But, by definition

$$\delta u(t) = \sum_i \frac{\partial u}{\partial p_i} \delta p_i \quad i = 1, \cdots, m \qquad \text{(A2-8)}$$

Substituting (A2-8) in (A2-7) and dividing throughout by δp_i, and neglecting higher-order terms, we get the ith component of $\langle \text{grad } J \rangle$ as

$$\langle \text{grad } J \rangle_{p_i} = 2 \int_R dR \int_{t_0}^{t_1} \frac{\partial \dot{u}(t - \tau)}{\partial p_i} \cdot \int_{t_0}^{t_1} \dot{u}(t - \tau) R_{\phi\phi}(\tau - \sigma) \, d\tau \, d\sigma - 2 \int_R dR \cdot \int_{t_0}^{t_1} \int_{t_0}^{t_1} \frac{\partial \dot{u}(t - \tau)}{\partial p_i} R_{\phi Q}(t - \tau) \, d\tau \, d\sigma \qquad \text{(A2-9)}$$

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