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ON THE SYSTEMS APPROACH IN HYDROLOGY

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ABSTRACT

There is a long felt need for a general theoretic frame work for hydrologic systems research which imbeds dynamical, structural, spatial and behavioral aspects of modeling with predictive powers. Many of these problems can be effectively studied by systems analysis. The process of system modeling for large-scale, nonlinear, time-lag systems can be rationalized by suitably identifying and modeling subsystems. When computers are used as modeling tools methods of formulating the problem, choice of the computer used, and choice of performance criteria greatly influence the results. This in turn imposes certain limits on the validity of computer simulated models. These aspects are discussed by studying the nature of hydrologic systems and the nature of the associated inverse problems and requirements for validating the models.

INTRODUCTION

During the past decade study of the hydrologic cycle and its various components has undergone a substantial change. At the turn of this century most of the research was confined to a delineation of the components of the hydrologic cycle and a descriptive discussion of the intervening phenomena. Since the 1930's, however, some significant advances were made in the quantification of hydrologic information and a great deal of work was done in this area by many eminent hydrologists.

During the current decade, there has been a considerable upsurge of activity in a systematic study of the theoretical and computational aspects of hydrological problems. Incidentally, this period coincided with the International Hydrological Decade. This world-wide activity is partly responsible for a movement to upgrade the status of hydrology and relieve the empiricism involved in it. It is not an exaggeration to say that this goal has been achieved, at least in part. The fact that many mathematicians and engineers from other disciplines are currently engaged in research activities that traditionally belonged to the hydrologist speaks for itself the changing nature of problems that hydrology can offer. Due to this widened interest in hydrology, from without, the subject has been flooded by numerous papers. In particular, the concept of a system, borrowed from electrical engineering, found widespread use by the hydrologist. The deluge of material written on system theory (mostly by electrical engineers and mathematicians) concerns general theorems but very little of usable techniques for obtaining practical results. Early attempts to bring this new tool to the aid of the hydrologist also caused some problems of communication which were accentuated by a proliferation of alien terminology. This resulted in a gap—a gap due to a lack of understanding of the meaning of a system as it was conceived and developed by its originators. This gap is widening as new generations of hydrologists bring in new tools of research which are regarded so far as lying beyond the frontiers of interest of the traditional hydrologist.

So far, a couple of attempts have been made (Amorocho and Hart, 1964; Dooge, 1968) along the lines similar to the one taken in this paper. This paper complements the above two works. The presentation starts with some notions about a system and then attempts to classify the nature and role of various classes of problems that a systems engineer faces. Even though many illustrative examples from hydrology are cited, the expose is valid for any engineering system.

Nature of an engineering system

The essence of all engineering endeavors lies in the study of physical systems. Let us, for a moment, leave the concept of a system undefined and let us tacitly assume that the general nature of a system can be understood intuitively. A basic problem in natural and behavioral sciences is first to describe the behavior of a system in some convenient fashion, then use the description to predict the future behavior, and finally apply this clairvoyance in some useful way.

In mathematical parlance the goal is to determine the *state* of a system at some future time from a knowledge of its present or initial state. For example, we may wish to determine tomorrow's weather from today's weather conditions. To determine the behavior of the weather at a particular time in the future, generally we must be able to determine the weather at any time in the future. This is a formidable computational job and the problem of avoiding the generation of this proliferation of data is of fundamental importance.

The mathematician attempts to solve this problem by using a very simple device—by using differential equations. If $x(t)$ represents the state of a system at any time t, one can use a differential equation of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = g(x); \quad x(t=0) = x_0 \tag{1}$$

to obtain the value of $x(t)$ at all future times and therefore at any particular time. In the majority of cases, the basic equation is a nonlinear partial differential equation, or a functional equation more complicated in form than the ordinary differential equation shown above.

Hydrology is the science of studying the properties of water and its movement in various parts of the water cycle and the science of engineering and managing this water as a natural resource. If this definition is accepted, problems in hydrology can be divided into two classes: (1) The scientific or technological aspects of the problems and (2) the management or behavioral aspects of the problem. Even though science and technology are vital to the continued development of natural resources to serve mankind's needs, the social structure within which technology is applied is equally vital. Hydrological systems are some of those that have to operate within a constraining behavioral frame of reference and study of one aspect (say, the technological) without due regard to the other (behavioral) aspect will be meaningless.

Technological subsystem

Problems typical of environmental sciences are characterized by their large size and are responsible for a new genre of mathematical sciences—the study and control of large scale systems. A significant practical aspect of the problem here is not even a question of control; that is, far too ambitious. It is a question of learning enough about a system to permit the development of a meaningful policy for operation. The general problem of operating a large system with limited amount of time available for observation, data processing and implementation of control generates new kinds of mathematical questions that have not yet been precisely formulated and certainly not resolved.

Uncertainty in the measurement process is another factor of fundamental importance in the study of large scale systems. The problem of decision making in complex situations is meaningful only if we know how to deal with this element of uncertainty during the model-building phase.

A third problem of importance in techno-behavioral (technavioral?) sciences is the element of time scale. In many physical sciences, time periods for which mathematical functions apply are necessarily short whereas months, years and even decades are typical time units in a majority of problems in hydrosciences. Either the systems are heavily overdamped (such as rainfall-runoff systems) or they have long time delays (delay between investment and benefit). Very often these time delays are functions of time making the governing equations very complex.

A fourth factor of mathematical significance is that the input (i.e., the exciting signals) is not generally under the control of the observer and sometimes may not even be directly observable.

Behavioral subsystem

One of the foremost areas of research of fundamental importance, and ironically the least explored, is the effect of behavioral structure within which a technological system operates. The reason for this situation may be a lack of understanding about the nature of problems in this area that are significant and susceptible for research. This observation is particularly true

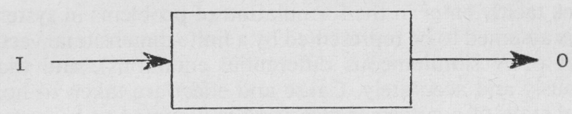

Fig. 1a

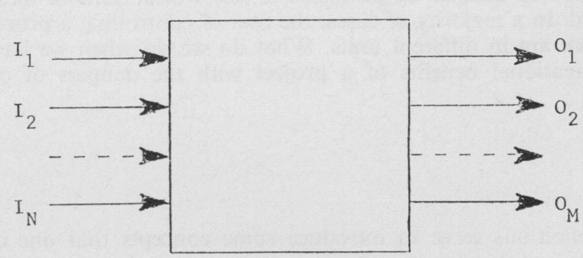

Fig. 1b

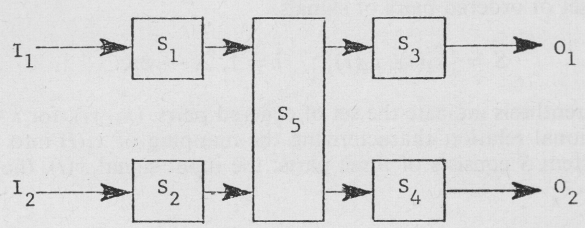

Fig. 1c

Fig. 1 — Configurations of a system.

in hydrology. The devious natural habits of water often ignore the jurisdictional boundaries created by man and pose the need for collective action of both public and private organizations. In fact, scientific and technological capability to handle water—water management needs— are almost powerless unless translated by effective and adequate institutional arrangements into significant social values.

Intuitively, a system is nothing but a collection of interacting components subject to various inputs and producing various outputs. It is often convenient to conceive of a system in terms of a block diagram—the "black-box" representation as shown in figure 1 a. Often the situation is more realistic if the black-box has a variety of inputs and outputs as shown in figure 1 b. In the study of atmospheric systems, geological, hydrological and other environmental systems, it is useful to think in terms of diagrams such as the one in figure 1 c which brings clearly the existence of internal subsystems whose behavior can neither be examined nor influenced directly.

The behavior of systems is not always exemplary: economic systems are subject to inflation and depression; hydrological systems are subject to, say, floods; ecological systems are subject to pests and drought. One of the main tasks of an engineer is to control the behavior of systems and to derive benefits from his ability to control.

Tacit assumptions and uncertainty

Many assumptions tacitly enter in the formulation of problems in system theory. Generally the state of a system is assumed to be represented by a finite dimensional vector (i.e., equation (1) now represents a set of N simultaneous differential equations), and that the state can be observed instantaneously and accurately. Cause and effect are taken to hold and moreover to be known. The initial state of a system is also usually assumed to be known and the effect of the propagation of errors due to inaccurate initial conditions is neglected. Finally, and paradoxically, one starts a problem with the assumption that the objective or goal in studying the process is known and well defined. In problems of real world, none of these assumptions are verified and validated. In a majority of cases, the cost of controlling a process and the reward for good performance are in different units. What do we do when we have to compare the economical and recreational benefits of a project with the dangers of cultural and social dislocations?

Useful definitions

The following definitions serve to introduce some concepts that one commonly uses in system theory and are useful in the sequel.

System

A system S is a set of ordered pairs of signals.

$$S = \{x_i(t), y_i(t)\}; \quad i = 1, 2, \cdots \text{etc.} \tag{2}$$

where the braced parenthesis indicate the set of ordered pairs. (x_i, y_i), for $i = 1, 2, \ldots$ etc. with an underlying functional relation characterizing the mapping of $x_i(t)$ into $y_i(t)$.

Essentially, a system S consists of three parts: the input signal $x(t)$, the output signal $y(t)$ and the plant P.

Input

The members of the *domain* of S, i.e., the set $\{x_i(t)\}$ are said to be the inputs to the plant P.

Output

The members of the *range* of S, i.e., the set $\{y_i(t)\}$ constitute the output of the plant P.

Plant

The plant P of a system S contains a mathematical characterization of the process that relates the inputs and outputs.

This distinction between a system S and its plant P is very important and many people erroneously regard the terms "system" and "plant" as synonymous. A plant together with its inputs and outputs constitute a system. To make this distinction more meaningful, any physical entity whose behavior is required to be controlled may be regarded as a plant. A combination of plant(s) or components that act together and perform a certain function may then be regarded as a system. However free use of these two terms interchangeably is quite common in the literature with no ambiguity to the discerning eye.

Charaterization

A plant P of a system S is said to be characterized by a differential operator (or an integral operator, an integro-differential operator or a functional operator—just to mention a few), an auxiliary condition (such as an initial condition) if and only if each signal pair $(x(t), y(t)) \in S$ provides a solution to the associated equation and the auxiliary condition and each such signal pair is a member of S.

That is, a system (or plant's) characterization completely describes its input/output relation.

Function

A function $f(.)$ is a single valued set of ordered pairs $\{(x_i, y_i)\}$, $i = 1, 2, \ldots$ etc., i.e.,

$$f(.) = \{(x_1, y_1), (x_2, y_2), \cdots, (x_N, y_N)\} \tag{3}$$

A signal as defined above may be regarded either as a function of time or as a set of ordered pairs.

Functional

A functional $F[.]$ is a single valued set of ordered pairs of scalar valued functions of one real variable, $f_i(.)$ and scalars z_j, i.e.,

$$F[.] = \{(f_k, z_k)\} \tag{4}$$

Whereas a function defines a mapping from one point to another point, a functional defines a mapping from a function to a point. A simple example of a functional is the pair

$$\left(g(\xi), \int_a^b g(\xi)\,\mathrm{d}\xi \right),$$

where a mapping from the function $g(\xi)$ to the point whose value is defined by the definite integral $\int_a^b g(\xi)\,\mathrm{d}\xi$ is implied.

By letting

$$z = \int_a^b g(\xi)\,\mathrm{d}\xi \tag{5}$$

a functional can be written in an alternate form

$$z = F[g(\xi)]_{\xi=a}^{\xi=b} \tag{6}$$

Heuristic

A heuristic (method) is a rule of thumb, strategy, trick, simplification, or any other kind of device which drastically limits the effort in search for solutions of problems. All that can be said about a useful heuristic is that it offers solutions which are good enough most of the time.

Algorithm

An algorithm (or algorithmic methods) is a decision procedure which guarantees the solution sought given enough time.

A variety of physical problems can be studied effectively by adopting the systems point of view. A basic advantage of such an approach is in the possibility for standardization of analytical and computational techniques. Once a handful of techniques are mastered, it becomes only necessary to extend the arguments to be applicable to any physical system. Furthermore, the systems representation helps to identify analogous situations which may otherwise escape attention. The extent to which systems approach can be profitably employed becomes clear by considering as an example the task of developing a generalized planning system to predict and study the future water requirements of a community. Various facets of such an effort are diagrammatically shown in figure 2. From this figure it is clear that the social structure within which science and technology are applied is as vital as the technology itself. Modernization of water law, political institutions related to water resources development and management and assessment of existing water resource policies are but a few areas where meaningful contributions can be made by using the systems approach.

Nature of engineering problems

Problems of interest to a systems engineer can be broadly classified as *direct* and *inverse* problems. Direct problems are characterized by a complete specification of the contents of the box in figure 1a and one is required to study or predict the response of the box to any specified input. This problem is also called the *analysis* problem. From a mathematical point of view an analysis problem, in general, constitutes the solution of a differential equation, integral, integro-differential or some such functional equation.

In engineering *analysis*, the system and the excitation are specified and the response is to be found. The precise nature of the excitation and response depends, of course, on the physical area to which the system belongs. In the case of a distributed system, a field, the specification must include both the distributed characterics of all points within the field and the geometric location of the field boundaries.

The inverse problem is much more complex. Here the response to a particular input or inputs is known, but either the equations describing the process are unknown, or the inputs themselves are unknown. Depending upon the specific circumstance, an inverse problem may present itself as (*a*) *design* or *synthesis* problem, (*b*) *identification* or modeling problem, and (*c*) *instrumentation* or *control* problem. While the direct problem generally has a unique solution, if it has any solution at all, the same is not true for an inverse problem. An inverse problem often leads to a multitude of mathematically acceptable solutions out of which one physically acceptable solution, if any such solution exists, has to be selected. Criteria for this selection depend upon the nature of the inverse problem.

In a design or synthesis problem, the nature of the expected excitation and the nature of the required response are specified and a system having this excitation-response relationship has to be physically realized or designed as it is often called. For example, it is possible to have many possible alternatives in the design of a penstock or surge tank that satisfies some stipulated

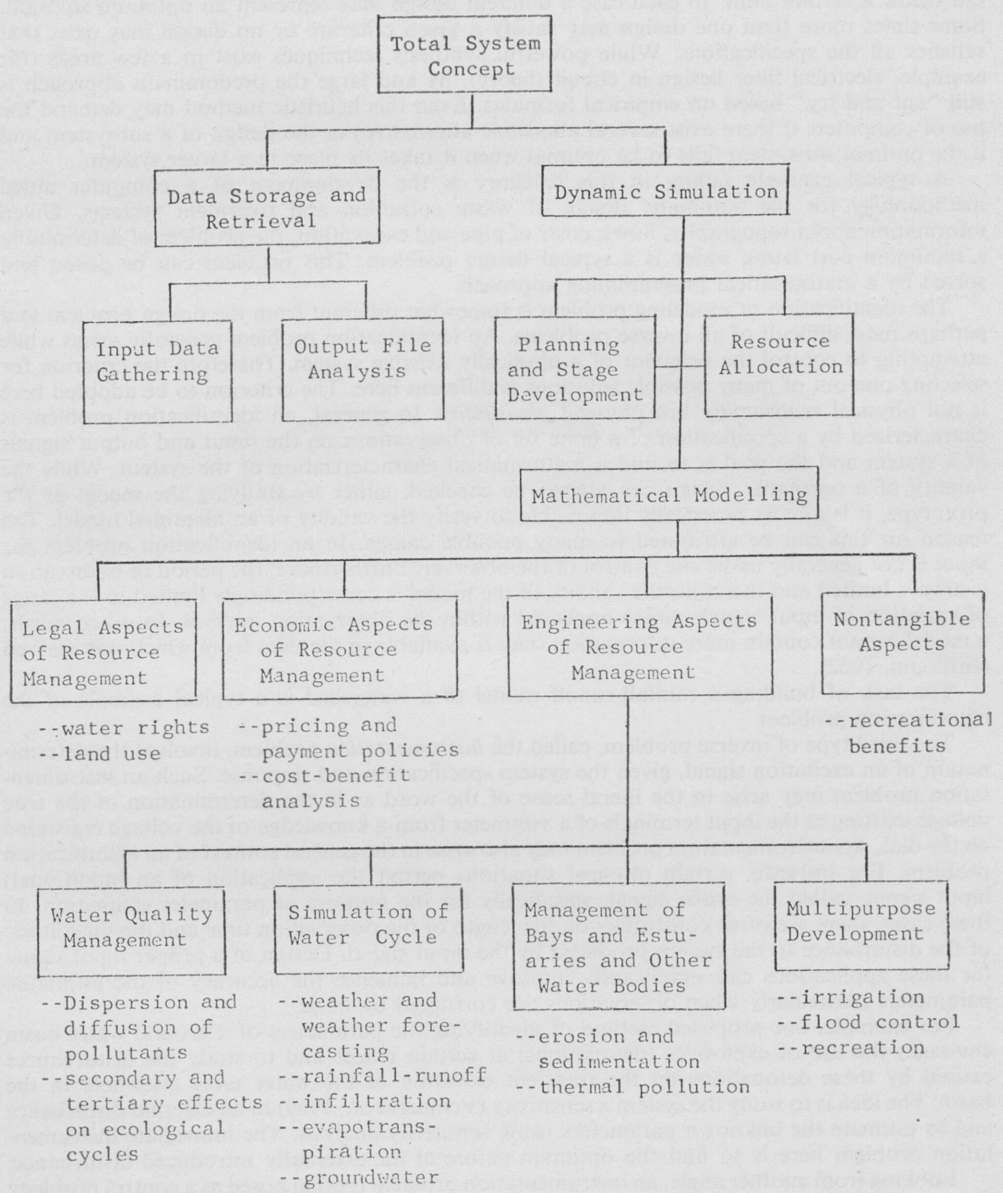

Fig. 2—Nature of hydrological problems that fall within the system concept.

performance requirements. To choose one out of many possibilities it is necessary to establish design criteria or design specifications. Perhaps it is necessary to minimize the cost, perhaps it is necessary to meet the peak load demands, perhaps the drop in hydraulic head should not fall below a certain limit. In each case a different design may represent an optimum solution. Some times more than one design may satisfy a given criterion or no design may exist that satisfies all the specifications. While powerful synthesis techniques exist in a few areas (for example, electrical filter design in circuit theory), by and large the predominant approach is still "cut and try" based on empirical formulas. Even this heuristic method may demand the use of computers if there exist several allowable alternatives in the design of a subsystem and if the optimal subsystem fails to be optimal when it takes its place in a larger system.

A typical example falling in this category is the development of a computer aided methodology for the automatic design of waste collection and treatment systems. Given information about topography, flows, costs of pipe and excavation, the problem of determining a minimum cost trunk sewer is a typical design problem. This problem can be posed and solved by a mathematical programming approach.

The identification or modeling problem is somewhat different from the design problem and perhaps most difficult of all inverse problems. An identification problem generally arises while attempting to control the behavior of a physically *existing* system. Therefore the criterion for selecting one out of many possible solutions is different here. The criterion to be adopted here is not physical *realizability* but physical *plausibility*. In general, an identification problem is characterized by a specification of a finite set of observations on the input and output signals of a system and the goal is to find a mathematical characterization of the system. While the validity of a particular design can always be checked, either by studying the model or the prototype, it is almost practically impossible to verify the validity of an identified model. The reason for this can be attributed to many possible causes. In an identification problem the input is not generally under the control of the observer. Furthermore, the period of observation is always limited and therefore the validity of the model is correspondingly limited to the range of variation of input signals which appeared within the observation interval. In other words, a model cannot contain more information than is available in the data from which it is derived (Brillouin, 1962).

The task of building a rainfall-runoff model to a watershed is a typical example of the identification problem.

The third type of inverse problem, called the *instrumentation* problem, involves the determination of an excitation signal, given the system specification and response. Such an instrumentation problem may arise in the literal sense of the word as in the determination of the true voltage existing at the input terminals of a voltmeter from a knowledge of the voltage registered on the dial. An instrumentation problem may also arise in the general context of an identification problem. For instance, certain physical situations permit the application of an (additional) input signal, called the probe signal, specifically for the purpose of parameter estimation. In these cases, there are often constraints on the length of the observation time and the magnitude of the disturbance to the system produced by the input signal. Design of a proper input signal for these applications can significantly improve and influence the accuracy of the estimated parameters particularly when observations are corrupted by noise.

For example, one proposed method of identifying the parameters of a ground water basin envisages the use of explosives (detonations) at certain places and to study the disturbances caused by these detonations on the transient elevation of the water table elsewhere in the basin. The idea is to study the system's sensitivity (Vemuri *et al.*, 1969) to an external disturbance and to estimate the unknown parameters using sensitivity analysis. The immediate instrumentation problem here is to find the optimum nature of the externally introduced disturbance.

Looking from another angle, an instrumentation problem is often posed as a control problem. In a control problem one is interested in the determination of the characteristics of an external signal that drives a system from a given initial state to a desired terminal state. This control problem formulation has received a great deal of attention and all other aspects of an inverse problem have been studied in the general context of a control problem.

Essential elements of a control problem can be listed as follows:

(*a*) A mathematical model of the system to be controlled;

(*b*) A description of the desired output of the system;

(*c*) A set of admissible inputs or controls;

(*d*) A performance or cost functional which measures the effectiveness of a given control criterion.

A typical example of a control problem in hydrology is the task of determining a pumping/recharge *policy* towards the optimum management of a ground water basin.

More about the identification problem

Characterization of a system and identification of a system are two fundamental problems of system theory. Broadly speaking, characterization is concerned with the setting up of various classes of mathematical models of physical systems. There are two principal lines of attack which are in vogue to characterize a system: the dynamical (or differential) equation approach and the kernel (or the weighting) function approach.

In the dynamical equation approach the unknown plant is assumed to be characterized by a known set of dynamic (very often, differential) equations of a given order but with unknown (and very often, constant) coefficients. This assumption regarding knowledge of the form of the differential equations is, of course, restrictive. Nevertheless, in practice, this information can often be gained from physical considerations. State variable and transfer function methods belong to this class.

The kernel function approach starts without making any assumptions of linearity or of dynamical equations of known form. The problem is thus nonparametric. A characterization is sought entirely in the time domain. In this approach, the unknown system (or plant) is characterized by an analytic function over a function space and represented by the associated power series (also known as Volterra) expansion (Balakrishnan, 1963). Identification of impulse response (unit hydrograph) essentially belongs to this class.

Whatever may be the method adopted to characterize a system, the computational process involved in solving the identification problem can be represented pictorially as shown in figure 3. It is evident from this figure that the problem itself divides into three parts.

(*a*) Determination of the form of the model (i.e., selection of the model dynamic equations) and isolation of the unknown parameters.

(*b*) Selection of a criterion function by means of which the "goodness of fit" of the model responses to the actual system responses can be evaluated.

(*c*) Selection of an algorithm for adjustment of the parameters in such a way that the difference between model and system responses, as measured by the criterion of (*b*) above, are minimized.

Selection of the model

The particular model (or class of models) that is chosen in a given situation depends upon a variety of considerations. Some of them are briefly summarized below.

a) Input

The input signal may be deterministic or random. Some times, the input signal may consist of a naturally occurring process added with a probe signal—the latter being intentionally introduced by the designer. Proper design of this probe signal belongs to the area of instrumentation referred to in the previous section. Some models can be theoretically identified only

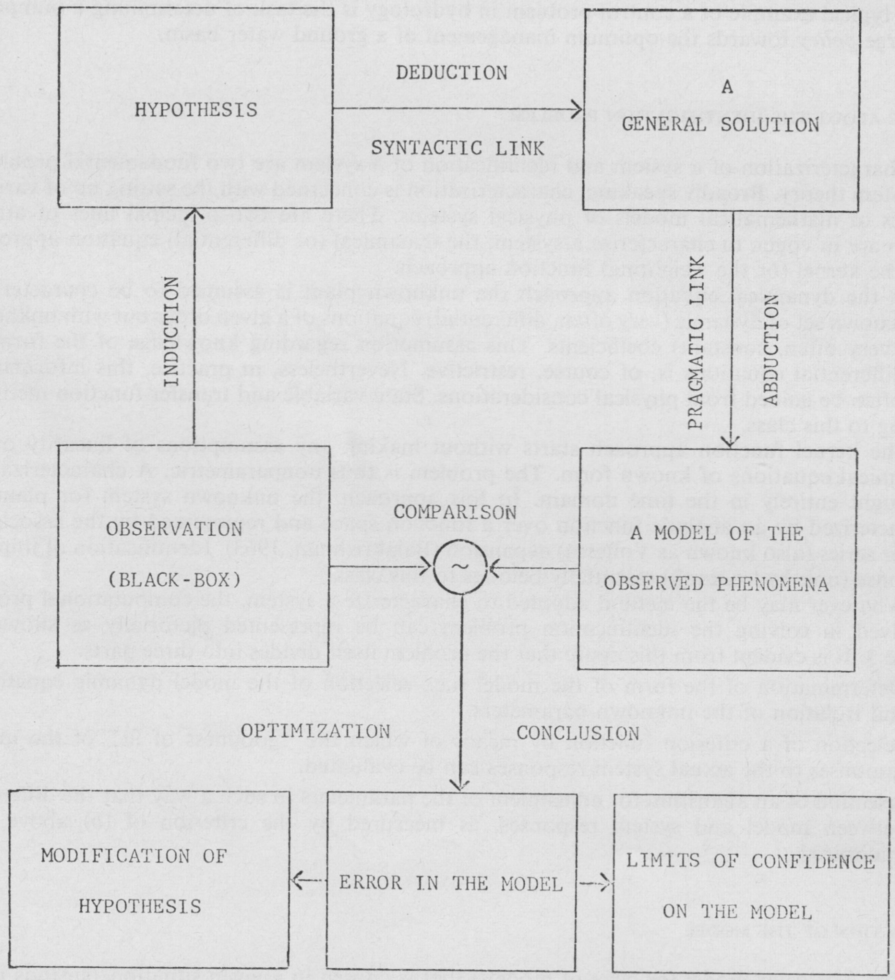

Fig. 3 — A block diagram of the problem solving procedure

when the input is white gaussian noise. If the actual input process does not meet this requirement, such a model cannot be used for identification purposes. However, on some occasions, a white gaussian noise which is uncorrelated with the actual input may be used as a probe signal.

b) In service/out of service

The kinds of inputs the system will be subjected to and the models that can be used, depend to a great extent on whether the system is in service or not. If the plant can be put out of service, at least temporarily, almost any input can be used and hence any theoretically acceptable model would be adequate. In some naturally occurring processes, it is almost impossible to put the plant out of service and the identification has to be carried out with the actual input without interrupting the process. Identification and control of such a process has to be done "on-line".

c) Black box versus gray box

A truly black box situation is one in which nothing is known about the plant or the nature of the noise corrupting the output signal. In general, the engineer has some a priori knowledge of the system and so the black box is really a gray box. It is evident that both black-box and gray-box modeling problems can be divided into three parts as listed earlier.

d) Stable and unstable plants

Certain processes are inherently unstable and many identification algorithms assume a stable plant. It is not true that unstable systems are useless systems. The practical problems involved in the identification of an unstable system are more complex. Uncontrollable floods can be regarded as a kind of instability in a river system.

In addition to the above considerations, the models chosen to characterize a system must be simple, adequate, easily modifiable and theoretically identifiable.

Selection of performance criterion

Formulation of a problem and selection of a particular method to characterize a system constitute only a first phase in the total problem solving effort. The task of approximation and computation is an equally important phase. Whatever the category of the particular problem may be, the first step in the general formulation of a computational strategy is the selection of an *objective*. That is, we set some goal to be achieved by our process or system through the application of a properly selected influence policy. Usually an objective is specified as the acquisition of some desired state for the process. A *constraint* is a limit or a required characteristic deliberately imposed on the system for any reason. One question that naturally arises in this connection is whether or not means for influencing the process under the imposed constraints are sufficiently strong to allow the achievement of the specified objective (controllability). If such means exist, then we have a properly formulated problem.

In general, there exist a number of ways in which the objective may be achieved. Within the set of possibilities, taking into account all the imposed constraints, one may wish to choose systematically the best approach with respect to some *performance criterion*, *cost function* or *penalty function*, as it is variously called. The nature of the criterion chosen, that is the method of measuring the "goodness" of a policy, obviously depends upon the particular system under study. Some of the general considerations useful in measuring the goodness of a policy are

1. Transfers of state must take place within a reasonable period of time.
2. The energy necessary to effect the changes in state must be constrained.
3. The expenditure of power required to effect changes must be constrained.

Consider, for example, the problem of selecting two parameters K and τ in order to optimize the performance of a string of pumping wells in a ground water basin. The parameter K could characterize the pumping "thrust" and τ some other parameter. In this context, it is conceivable to define optimum performance as that one which will minimize the total energy consumption and the time T required to achieve a desired state. This performance criterion can be written as

$$J = k_1 T + k_2 \int_0^T [\mathrm{e}(t, \tau, K)]^2 \, \mathrm{d}t \tag{7}$$

where $e(t, \tau, K)$ represents the deviation of the computed water level $h(K, \tau, t)$ from a desired level $h_D(t)$ and k_1 and k_2 are some constants. For each set of parameter values K and τ, J assumes a new value. The goal of the optimization procedures is to seek computational methods for automatically adjusting K and τ in order to minimize J. In certain cases it is desirable to find a time function, such as a pumping modulation program $K(t)$, rather than a fixed value of K. In ground water studies, this latter situation usually arises when one is interested in determining the "best" pumping program. In such cases, the problem becomes one of *functional optimization.*

There is a wide choice of criterion functions available. A particular function will be chosen, over the others, on the basis of mathematical and engineering considerations. Quadratic functions, such as the *integral square error* criterion

$$J_{ise}(h_D, h) = \int_0^T [h_D(t) - h(K, \tau, t)]^2 \, \mathrm{d}t, \tag{8}$$

are usually chosen because of certain mathematical advantages. When using a particular criterion function, it is important to understand its properties. For example, J_{ise} in equation (8) is a *functional in the output space* since it depends on the desired water table elevation $h_D(t)$ and the computed value $h(K, \tau, t)$. However, it is an *ordinary function of the parameters.* Therefore, insofar as the parameter optimization (or parameter identification, as it is often called) is concerned, use of the above criterion allows solution to proceed as in the minimization (or maximization) problem of ordinary calculus which is concerned with functions, rather than as in the problem of the calculus of variations which is concerned with functionals. This is an extremely important point to remember.

It is not always necessary, nor is it desirable, to choose the integral square error criterion. This criterion gives a measure of the deviation of the actual water table elevation from the desired elevation integrated over the entire duration of observation. At the end of a particular experiment, this criterion gives a number which will be large if errors persist over an interval of time. Indeed, any function which satisfies the requirements of a distance in the appropriate space can be used as a performance criterion. For example,

$$J_{ms} = \lim_{T \to \infty} \left[\frac{1}{2T} \int_{-T}^{T} (h_D(t) - h(K, t, \tau))^2 \, \mathrm{d}t \right] \tag{9}$$

$$J_{iae} = \int_0^T |h_D(t) - h(K, t, \tau)| \, \mathrm{d}t \tag{10}$$

$$J_{\sup} = \sup_{K, \tau} |h_D(t) - h(K, t, \tau)| \tag{11}$$

are all equally valid performance criteria.

The *mean square error* criterion, J_{ms}, is relatively easy to handle mathematically. It applies and is particularly useful when input(s) are statistical in nature. On the debit side, J_{ms} is useful only when the system is stable with bounded inputs. System identification based on the J_{ms} criterion often leads to lightly damped higher order systems. The J_{ms} criterion penalizes large errors much more severely than small ones and is relatively insensitive to parameter changes. Owing to this fact, the J_{ms} criterion is not highly recommended for use in parameter identification procedures. However, while solving a design problem, relative to a particular performance criterion, the system should be designed so as to minimize degradation of performance (criterion) due to parameter variations and under these conditions J_{ms} is highly useful.

The integral square error criterion, J_{ise}, has a close resemblance to J_{ms} and essentially has the same advantages and disadvantages as J_{ms}.

A performance criterion which provides increased sensitivity is the *integral of the absolute value of the error* J_{iae}. This criterion weighs large errors less heavily and small errors more heavily than J_{ise}. However, this is computationally more difficult to implement.

There exist a variety of tailor-made criteria to suit a given situation. In the case of static optimization problems, it is possible to define an instantaneous criterion function, such as

$$J(h_D, h(t)) = [h_D(t) - h(K, t, \tau)]^2 \tag{12}$$

rather than one which depends on the integration over a fixed interval. This kind of optimization can be carried out continuously since the effect of a change in the parameter is reflected immediately in a change in the criterion function. Static optimization problems of this kind are excellently suited for analog computer solution.

In dynamic optimization problems, evaluation of the performance criterion, in itself, constitutes a major computational process. If the minimization of J is based on the method of gradients, then evaluation of J not only requires a knowledge of the solution of the dynamic equation of the system but also that of an additional one called the derived equation. For example, in the sensitivity analysis method (Vemuri *et al.*, 1969), this additional equation is called the sensitivity equation which is always linear and posed as an initial-value problem. If dynamic programming method is used, this derived equation is the adjoint equation which is posed as a final-value problem (Vemuri and Karplus, 1969).

Selection of an algorithm

Techniques useful in the minimization of J (i.e., fitting of model to a given set of data) fall in several classes.

1. *Analytical Methods*

In some cases, analytical expressions for the unknown parameters can be derived. This is true, for example, for least squares estimates with certain assumptions on the nature of disturbance or noise entering figure 1.

2. *Search Techniques* (Wilde, 1964)

Are useful when the number of unknown parameters is small. In such cases, successive values of parameters are selected, either at random or in accordance with some preselected grid pattern in the parameter space and the corresponding values of J become $J^{(1)}, J^{(2)}, \ldots$ A simple comparison test is used to determine which value of J is a minimum.

Search techniques can be further subdivided into the following categories.

a) *"Brute Force" Methods*

The most obvious method of finding the optimal set of parameters is to discretize all the parameters and compute J for all combinations of these parameters. This method is also known as "exhaustive enumeration". The minimum value of J (and the corresponding parameter values) are then selected from the output listing. It is evident that such methods are useful only when the number of possible parameter combinations is low.

b) Elimination Methods

Elimination methods attempt to eliminate exhaustive search by confining attention only to a preselected part of the criterion surface. If the minimization procedure is visualized as a mountain climbing operation, where the mountain is assumed to have a single peak, then one can eliminate sections of the surface from further consideration by learning from the results of previous search. A particularly interesting elimination technique based on the principle of dynamic programming, is known as Fibonacci search (Wilde and Beightler, 1967).

c) Random Search

Another alternative to exhaustive enumeration is the use of random search in parameter space (Brooks, 1958). In this case, successive trial parameter values are selected at random. This method works even if the criterion surface has multiple minima. However, the number of explorations needed to insure a sufficiently high probability of obtaining a value near the true minimum may be extremely large. This method was used successfully (Hufschmidt, 1962) in the design of a water resource system.

Since search techniques require repeated solution of the model equations, they are well suited for hybrid computers (Bekey and Karplus, 1968), where rapid solution of differential equations is performed on the analog portion while programmed control and supervision of the computations remain in the digital part. Random search techniques are particularly useful as a method of optimizing nonlinear system parameters.

3. *Hill-Climbing Methods*

Rather than searching over the whole range of parameters, the various climbing or descending methods are based on finding the local properties of the criterion surface (Wilde, 1964; Dawdy and O'Donnell, 1965).

a) Relaxation Methods

This method is based on searching for the minimum along one parameter, finding the local minimum, setting the parameter at this value, continuing the search along the second parameter and so forth.

b) Gradient Methods

This is a direct approach to satisfy the system differential equations and constraints while iterating on the control signals until each new iterate drives the value of the cost function to a new minimum. If there are several local minima, gradient methods may have to be used repeatedly with different initial parameter values until a global minimum is found (Bryson and Denham, 1962; Vemuri and Karplus, 1968).

The main advantage of the method of gradients is the relative independence of convergence on the initial estimates of the unknown quantities. As the minimum is approached, the magnitude of the gradients decrease and correspondingly, the convergence becomes slower. Such techniques as quasilinearization (Bellman and Kalaba, 1965) and stochastic approximation (Kiefer and Wolfowitz, 1952) may be viewed as extensions of the gradient methods.

Special methods for linear systems

In those special cases when the system can be assumed to be linear, a number of specialized techniques are available for solving the identification problem. Consider, for example, a single input, single output constant coefficient linear system whose state at any time t is represented by the solution of the vector differential equation

$$\dot{x} = Ax + Bu; \quad x(t=0) = x(t_0) = x_0 \tag{13}$$

Solution of equation (13) can be written as

$$x(t) = \Phi(t-t_0)\,x(t_0) + \int_{t_0}^{t} \Phi(t-\tau)\cdot B(t)\,u(t)\,\mathrm{d}\tau \tag{14}$$

where $\Phi(t-t_0)$ is the system transition matrix, defined by

$$\Phi(t-t_0) = \mathrm{e}^{A(t-t_0)} \tag{15}$$

The identification problem of linear systems consists of estimating the elements of the matrix A or equivalently the transition matrix Φ from noisy measurements on the system response Y given by

$$Y = Cx + Du \tag{16}$$

We shall examine some of these techniques.

a) Weighting function identification

For simplicity, consider a scalar case of equation (13) in which the response can be described by the convolution integral

$$x(t) = \int_0^{\infty} h(t-\tau)\,u(\tau)\,\mathrm{d}\tau \tag{17}$$

where $h(t)$ is the system weighting function. If the input is a unit impulse, i.e., $u(t) \doteq \delta(t)$, then

$$x(t) = \int_0^{t} h(t-\tau)\cdot\delta(\tau)\,\mathrm{d}\tau = h(t) \tag{18}$$

Hence, the weighting function can be obtained, at least in principle, by measuring the impulse response of a system. In practice it is not possible to create an ideal impulse, so, a unit step response

$$x(t) = \int_0^{t} h(\tau)\,\mathrm{d}\tau \tag{19}$$

is first obtained and differentiation on both sides yields $h(t)$, the impulse response.

The unit hydrograph (Sherman, 1932; Nash, 1957, 1959, 1960; and Dooge, 1959) is essentially a weighting function of a watershed. Even though this method proved to be an excellent tool, assumptions regarding linearity of the watershed and the impulse nature of the exciting function on which the theory is based are almost always violated in practice.

An alternate approach to the "continuous" unit hydrograph method is the discrete unit hydrograph method. Because it is more practicable to record rainfall-runoff data at discrete intervals, equation (18) can be rewritten in discrete form as

$$x(nT) = \sum_{k=0}^{n} h(nT-kT)\,u(kT) \tag{20}$$

which yields

$$\begin{aligned} x(0) &= h(0)\,u(0) \\ x(T) &= h(T)\,u(0) + h(0)\,u(T) \\ x(2T) &= h(2T)\,u(0) + h(T)\,u(T) + h(0)\,u(2T) \end{aligned} \tag{21}$$

from which the following recursive relations can be derived

$$h(T) \quad = \frac{1}{u(0)} \cdot [x(T) - u(T)\, h(0)]$$

$$h(2T) = \frac{1}{u(0)} \quad [x(2T) - u(T)\, h(T) - u(2T)\, h(0)] \tag{22}$$

etc.

This procedure is called "numerical deconvolution" and yields satisfactory results in many cases.

b) Correlation Methods

If enough data is available permitting statistical evaluation, the weighting function (or unit hydrograph) may also be obtained from correlation measurements by using the Weiner-Lee relation

$$R_{ux}(\tau) = \int_{-\infty}^{\infty} h(t)\, R_{uu}(\tau - t)\, \mathrm{d}t$$

where $R_{uu}(\tau)$ represents the autocorrelation function of the input and $R_{ux}(\tau)$ represents the input-output cross correlation. In cases where large amounts of data are usually available, this statistical method is perhaps better than the unit hydrograph method.

c) Orthogonal Decomposition

In addition to the differential equation and weighting function representations an arbitrary linear system can also be modeled as a collection of filters whose impulse responses are orthogonal. (Mishkin and Braun, 1961; Dooge, 1968). That is

$$h(t) = \sum_{i=1}^{N} C_i \varphi_i(t)$$

where φ_i form an orthonormal set. The advantage of such a choice is that the adjustment of each parameter C_i is now independent of all others. If $h(t)$ is model response and $z(t)$, the observed response of the system, then the error is

$$e(t) = \sum_{i=1}^{N} C_i \varphi_i(t) - z(t)$$

If the mean square error is minimized, it is possible to compute the optimum values of C_i in a single computer run.

d) Transfer Function Methods

A third method is based on generating the impulse response of a linear system by way of functional transformations such as Fourier and Laplace transforms. Methods based on the use of moments (Nash, 1959), harmonic coefficients (O'Donnell, 1960) and Laguerre coefficients (Dooge, 1964) belong to this class.

METHODS FOR NONLINEAR SYSTEMS

Experience indicates that most of the component processes in hydrology are nonlinear in nature.

In general, any method available for the representation of a nonlinear functional can be chosen as a mathematical model for the identification of a nonlinear system. Some of the well known models available to characterize a nonlinear system are:

a) State vector representation;

b) Wiener model using orthogonal expansions;

c) The functional power series.

When a nonlinear system is represented by a state vector model, such as

$$\dot{x} = f(x, \alpha, t)$$

where both the order of the system as well as the form of $f(x)$ are unknown, the state variable representation has no advantage over the others. However, in a more tractable problem in which the form of the dynamic equations are known (as in equation 8) the value(s) of the unknown parameters can be estimated by using techniques based on quasilinearization, sensitivity analysis (Tomovic, 1962), numerical inversion of Laplace transforms (Bellman and Kalaba, 1966), in conjunction with such computational algorithms as the gradient descent method or mathematical programming.

Using an orthogonal development of nonlinear functionals in series of Fourier-Hermite functionals (Cameron and Martin, 1947), Wiener developed a model for the identification of nonlinear dynamical systems. In order for the approach to work, however, the input must be a white gaussian noise. In hydrology, most of the input processes are naturally occurring processes and therefore utility of this method appears to be limited.

In 1887 Volterra introduced the functional power series

$$\begin{aligned} r(t) = h_0 &+ \int_0^\infty h_1(\tau_1)\, p(t-\tau_1)\,\mathrm{d}\tau_1 \\ &+ \int_0^\infty \int_0^\infty h_2(\tau_1, \tau_2)\, p(t-\tau_1)\, p(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\ &+ \cdots \end{aligned} \tag{23}$$

for the expansion of the functional $r(.)$ in terms of various homogeneous functionals. It has been shown (Amorocho, 1963) that the linearity assumption is tantamount to truncating the above series after the second term. When Volterra series is used, as a mathematical model, the identification of a given nonlinear system reduces to the problem of determining the Volterra Kernels (i.e., the generalized impulse responses) $h_0, h_1, \ldots$, etc. Since h_i is a function of i variables, the determination of these functions for $i \geqslant 3$ becomes a formidable task. Some initial efforts to apply this technique were made by some hydrologists (Amorocho and Orlob, 1961).

A variation of the above method is to decompose the nonlinear time-lag system into two subsystems: one a linear time-lag subsystem and the other a nonlinear no-time-lag subsystem (Jacoby, 1966).

A basic disadvantage with this power series method lies in the difficulty to interpret the meaning of the functions h_i in terms of the physical parameters of a system but the method is general enough to be applicable both to linear and nonlinear systems. The transfer function approach has several advantages in linear cases and attempts to generalize this method to nonlinear systems (George, 1959) are not yet popular.

MODELS VERSUS REALITY

While solving scientific and engineering problems it is important to keep in mind distinction between theories and reality. Theoretical results are derived from certain axioms by using principles of deductive logic. In physical sciences, the theories are so formulated that they correspond, in some useful sense, to the real world. The separation between the conceptual world (model) and the physical world (reality) should be clearly imprinted in the analysts mind at all times.

It is important to remember that the word "model" is meant to imply a manifestation of the *interpretation* that the scientist gives to the observed facts. Facts remain unchanged, but models change. Thus the term "model" covers a vast variety of configurations. It can be a laboratory mechanism, such as a Hele-Shaw model or an electrolytic tank analog, it can be merely a system of mathematical equations for which a simple and straightforward laboratory tool cannot be easily devised (such as Maxwell's equations of electromagnetic field theory) and in the extreme case of abstraction, a model may exist only in the mind of a scientist (such as Bohr's model of an atom). Similarly, a model may be very modest in its aims just like the experiments it is expected to sum up.

NATURE OF SOLUTIONS SOUGHT

At this point it is instructive to make a distinction between scientific and engineering analysis. A mathematician, when confronted with a problem, worries about existence and uniqueness of solutions, validity of certain commutative multiplication or some such delicate question without concerning himself with the physical meaning of his operations. Mathematics is an edifice built on a set of axioms and as long as he conducts himself without violating the axioms, lemmas and theorems a mathematician would be satisfied. The interests of a scientist, of course, lie in the physical world; however, his attitude is more or less philosophical. The goal of a scientist is to construct theories to explain the observed behavior of the physical world around him and for this purpose he has to take into account all possible alternatives to explain certain phenomena. The goal of engineering analysis is to obtain specific answers to specific questions with a specified accuracy at a minimum cost in time, labor, and equipment. Moreover, while an engineer would certainly like to obtain general solutions, he is generally willing to settle for numerical solutions.

ROLE OF SIMULATION

One of the earliest methods known to man to study the nature of physical systems is by simulation. When confronted with the problem of studying the behavior of physical phenomena, the engineer or scientist constructs a scale model following Newton's Law of Similitude which states that models must be dynamically and geometrically similar to the structures that they represent. In its primitive sense, a model is a small scale representation of the prototype in all features which are pertinent to the problem under investigation. Often it is impossible to satisfy all the requirements of similitude simultaneously. In this event, skill is required to design the model in order to reduce the error by ignoring some of these requirements to a negligible value.

A major drawback of the method based on similitude is a lack of flexibility. Electrical analogs (Karplus, 1958) introduced an element of flexibility and played a very prominent role in the study of hydrologic systems. With improved sophistication in the available hardware, the term "simulation" also emerged with new meaning and depth. The word simulation now stands to mean trial and error experimentation in which the validity of a model is verified:

sensitivity to environment is explored; and variation of performance to parameter changes evaluated. With this broader meaning simulation has become a means of solving various facets of the inverse problems described earlier. Where optimal programming methods are not suitable because of the complexity of the model, simulation offers the possibility of using an experimental approach to the decision problem. When the models are very complicated with complex quantitative relationships that make explicit mathematical solution impossible or difficult, the use of modern computers in conjunction with numerical approximation methods offers a feasible alternative.

Validation of computer simulated models

Experimental verification of physical models is done by subjecting them to the characteristic signals that the physical system is likely to encounter and examining the response signals. Construction of scale models with precision and observing the results with precision is possible in case of systems occurring in the physical sciences. In the case of behavioral systems it is not easy to construct true models. If, for instance, the management of a ground water basin is influenced by factors such as socio-economic or political considerations, it is almost impossible to simulate the total environmental system by using scale models. Mathematical models with a greater degree of abstraction are used invariably to study the behavior of such systems. It is obvious therefore that the Law of Similitude in this respect would mean not the proportionality of parts but similarity of logical process in the behavior of the system and its model.

In problems of inference from model to the real system, the question arises whether the model under examination is a valid one, or to what extent can it be considered a valid model? In the field of hydrology, most of the efforts so far seem to have been directed in building complex models on computers and the problem of validation does not seem to have received much attention. Basically, the validation of a simulated model does not pose a problem different in principle from the validation of any other scientific hypothesis, but the complexity that is typically built into such models is so great that the process of validation is different.

Important features of a scientific method may be briefly summarized as follows. (1) Careful and accurate classification of facts and observation of their sequence and correlation; (2) Discovery of scientific laws by the aid of creative imagination; (3) Equal validity for all normally constituted minds; (4) Self criticism.

In order to test the validity of a theory or a model we may use these criteria by putting the following questions and trying to elicit positive answers. (1) Does the theory permit careful and accurate classification of facts and observance of their sequence and correlation? (2) Does the theory provide scope for discovery of scientific laws by creative imagination? (3) Is the theory equally valid for all normally constituted minds? (i.e., Are we sure that we will not encounter a "Maxwell's demon" as we extend the theory and try to generalize it?) (4) Is the theory or the model capable of withstanding criticism? If we get positive answers to these questions, it means that our theory has stood the four-fold test and is scientifically valid. Insofar as computer simualtion is concerned, we may say that it satisfies this four-fold test for: (1) Preparing the model for simulation means writing a suitable program which in turn involves processing the data classification and correlating the data. (2) Interpreting the computer results and predicting the behavior of the real system involves logical inference and creative imagination. (3) The logical mechanism of the computer ensures uniform results whosoever feeds the same data thus obviating the differences between normal and abnormal minds. (4) The final and the most important touchstone of validity of a scientific hypothesis is it's ability to sustain criticism. A scientific hypothesis must be capable of being disproved. Theories or models should be subjected to tests capable of showing them to be false. As Karl Popper (1931) said, "So long as a theory withstands detailed and severe tests and is not superseded by another theory... we may say that it has proved it's mettle or that it is corroborated."

Having come to the conclusion that a scientific hypothesis or model should be subjected to the most severe tests aimed at disproving it, we should discover the relevant tests that are most appropriate to the problem at hand. A physical model is sought to be validated by ensuring similitude in construction and verifying whether the changes in state produced in it are truly representative of the theoretically expected values of the real system. Validation of behavioral models is however not that easy because of the unpredictable behavior of the variables involved. The need to construct behavioral models in hydrology occurs because of the inherent interaction of scientific and technical variables with such factors as social, political, legal, and ethical considerations. When such factors enter into the equations of optimization, then the validity of the overall models can be judged from two points of view, namely, determining the truth value of the results obtained and from a utilitarian point of view.

Determination of the truth value involves verification at three stages to be sure; namely verification of the validity of the concept, validity of inference and verification of empirical concordance. Several philosophical theories have been in existence which discuss this problem of conceptual validity. Important among them are *rationalism*, *empiricism*, and *positivism*. Rationalism holds that a model or theory is simply a system of logical deductions from a series of synthetic premises of unquestionable truth values; not themselves open to verification. The phrase "synthetic a priori" has been coined by Kant (Russell, 1946) to describe the premises of this type. Thus, for the rationalists, the problem of verification of a theory reduces to the problem of searching for a set of basic assumptions underlying the behavior of the system under study. At the other end of the spectrum is empiricism. Empiricism refuses to admit any postulates or assumptions that cannot be independently verified. Empiricists ask that we begin with facts and not with assumptions.

Validity of Inference

Inference is scientifically valid if the inferred could be drawn by every logically trained mind and if the inference is drawn from known things to unknown things. Establishing the validity of inference in simulation problems is of paramount importance.

Experimental Verification

The final stage of verification is the experimental verification. In social sciences, one cannot always conduct experiments under controlled conditions and at a particular instant of time.

Experimental verification at this stage means testing how far the results predicted by simulation agree with actually observed values. Here one can use the non-parametric methods of statistical inference to test the correspondence between fact and theory and establish confidence intervals on the performance of computer simulated models. For instance, one can use the simple "chi square test" and decide whether there is any correspondence between observations and predictions at a particular probability level.

Perhaps the best method of verification may be subjecting the model to multistage verification: first, searching for Kant's synthetic a priori, secondly, attempting to verify the postulates subject to the limitations of statistical tests and finally, testing the truth value and ability of the model to forecast the behavior of the system under study.

Conclusions

1. Hydrologic systems, in a broad sense, consist of two (not necessarily distinct) subsystems interacting with each other. One of these subsystems deals with processes which are essentially hydrologic in nature and the other deals with behavioral aspects which enter the scene as a result of human intervention.

2. Hydrologic systems are, in general, large-scale, time-varying, nonlinear, time-lag systems. Variation with time may be due to human intervention (as in the effect of urbanization of a watershed) or may be due to such natural factors as seasonal variations.
3. The fact that inputs are not generally under the control of the observer and also the fact that precise measurements are not easy to come by introduces an element of uncertainty in the study and modeling of hydrologic systems. By cleverly establishing measures of uncertinty, it is possible to construct meaningful models and establish confidence intervals on the validity of the models obtained.
4. Where some physical insight into the operating process of a system is easy to obtain, the dynamic equation approach to model building is more useful; it gives an opportunity to relate macro- and micro-properties of the system. Where almost nothing is known about the physical process and nothing about the structure and order of the dynamic equations, non-parametrization is more meaningful.
5. Finally, if computers are used as model building tools, the algorithms and programs should be conceived and written with the following points in mind. (*a*) The programs should permit heuristic modification of an algorithm. (*b*) During simulation of a model, the program should permit for "on-line" ADD and DELETE instructions that permits the operator to substantially change the sequence of operations during the solution process. (*c*) The program should contain built-in check routines to check the validation of the computer output.

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